

New twist field couplings from the partition function for multiply wrapped D-branes

Dario Duò and Rodolfo Russo

*Centre for Research in String Theory, Department of Physics,
Queen Mary, University of London Mile End Road, London, E1 4NS, U.K.
E-mail: D.Duo@qmul.ac.uk, R.Russo@qmul.ac.uk*

Stefano Sciuto

*Dipartimento di Fisica Teorica, Università di Torino and INFN, Sezione di Torino,
via P. Giuria 1, I-10125 Torino, Italy
E-mail: Sciuto@to.infn.it*

ABSTRACT: We consider toroidal compactifications of bosonic string theory with particular regard to the phases (cocycles) necessary for a consistent definition of the vertex operators, the boundary states and the T-duality rules. We use these ingredients to compute the planar multi-loop partition function describing the interaction among magnetized or intersecting D-branes, also in presence of open string moduli. It turns out that unitarity in the open string channel crucially depends on the presence of the cocycles. We then focus on the 2-loop case and study the degeneration limit where this partition function is directly related to the tree-level 3-point correlators between twist fields. These correlators represent the main ingredient in the computation of Yukawa couplings and other terms in the effective action for D-brane phenomenological models. By factorizing the 2-loop partition function we are able to compute the 3-point couplings for abelian twist fields on generic non-factorized tori, thus generalizing previous expressions valid for the 2-torus.

KEYWORDS: D-branes, Bosonic Strings, Intersecting branes models.

Contents

1. Introduction	1
2. Toroidal compactifications	4
2.1 Some properties of T^{2d}	4
2.2 Closed strings on T^{2d}	5
2.3 The Boundary State for a wrapped magnetized D-Brane	8
2.3.1 Gauge bundles and gauge invariant Wilson loop	9
2.3.2 Wrapped D-Branes as non abelian gauge bundles	10
2.3.3 Computation of the Boundary State	12
3. The twisted partition function	15
4. Twist fields couplings on T^{2d}	20
4.1 The two-torus example	21
4.2 Twist fields couplings on a generic T^{2d} compactification	23
A. Generic and block-diagonal integer matrices	27

1. Introduction

The relation between the modular and the unitarity properties of open string amplitudes have played a crucial role in deepening our understanding of string theory. For instance, Lovelace [1] studied the modular transformation of 1-loop non-planar open string amplitudes. By requiring that these amplitudes do not contain cuts, he discovered the critical dimension of bosonic string theory. More than twenty years later Polchinski [2] used the same modular transformation between the open and the closed string channels on the 1-loop partition function with Neumann and Dirichlet boundary conditions. By doing so he was able to show that D-branes are actually dynamical objects that have gravitational couplings with closed strings.

The same interplay between modular transformations and unitarity properties exists also for higher loop amplitudes. For instance, the open string diagram for the 2-loop planar partition function is a Riemann surface with three boundaries and no handles. This surface can be described either in the closed string channel as in figure 1a, or in the open channel as in figure 1b. In the first case, by unitarity, one should be able to decompose the result into a 3-vertex among closed strings and three boundary states. In the open string parametrization, on the other hand, the same amplitude should factorize into three open strings propagators and two 3-point vertices among open strings. This

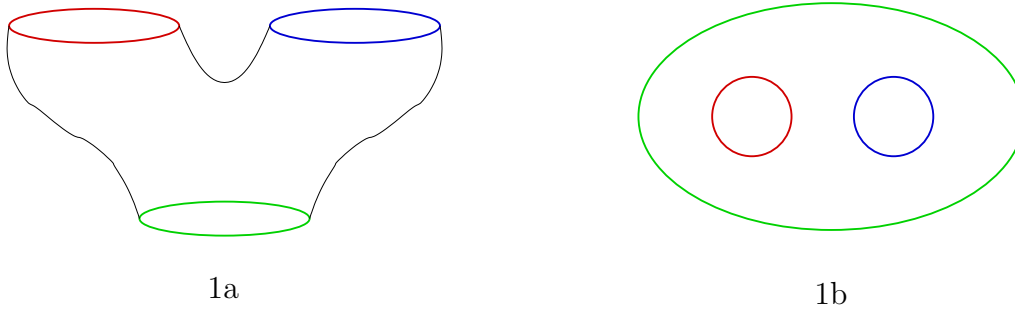


Figure 1: The twisted open string partition function with three borders and no handles (i.e. on each boundary different left/right gluing conditions are imposed). In 1a the amplitude is depicted in the closed string channel, while in 1b it is modular transformed and the open string propagation is manifest.

double description of the 2-loop partition function is particularly interesting when different left/right gluing conditions are imposed on the various boundaries. In [3] the bosonic contribution to the twisted g -loop partition function was computed by using the closed string description and then was modular transformed in the open string channel. It was also shown that the factorization of the 2-loop diagram provides an effective strategy to compute the couplings among twists fields (σ_ϵ), which is alternative to the stress-energy tensor technique of [4]. From the Conformal Field Theory (CFT) point of view, these σ_ϵ 's are operators implementing a change in the boundary conditions for the (complexified) bosonic coordinates. The boundary conditions induced by the σ_ϵ 's are appropriate to describe open strings stretched between D-branes with constant magnetic fields [5] or D-branes at angles [6]. This kind of open strings is one of the main ingredients in D-brane model building (for a recent review see [7] and the references therein). The 3-twist field correlators mentioned above provide the non-trivial part of the Yukawa couplings¹[12–16, 3] and of other terms in the effective action generated by stringy instantons [17, 18].

In this paper we generalize the results of [3] in various ways. We compute the bosonic contribution to the planar g -loop partition function for open strings on generic tori with both multiply wound D-branes and non-zero Wilson lines or open string moduli. Then we use this result to derive an explicit expression for the 3-twist field correlators valid beyond the case of 2-torus. However, in our computations, we still keep a technical assumption: the monodromy matrices characterizing the open strings stretched between the D-branes must commute, see eq. (3.1). This is always the case when the gauge field strengths on the various branes are themselves commuting or parallel. However this more stringent condition is not necessary and we are able to compute the partition function also for some setups involving oblique fluxes (according to the nomenclature of [19]). On the other hand the assumption (3.1) will restrict our final results on the twist field couplings to the abelian case. The D-brane configurations studied in this paper can be promoted to

¹Actually the twist field couplings play the same role also in phenomenologically interesting compactifications of Heterotic string theory [4, 8–11]

supersymmetry preserving setups, once they are embedded in superstring theory. This is achieved simply by imposing some constraints on the magnetic fluxes to satisfy the D-term and the F-term conditions. Moreover the latter ones are often automatically implied by our hypothesis (3.1), since it ensures that there is a particular complex basis for the torus where almost all magnetic fluxes are described by $(1, 1)$ -forms.

As it was done in [3], we compute the twisted partition function in the closed string channel by using the operator formalism (see [20, 21] for detailed discussion). This approach requires particular care in dealing with some phase factors present in the definition of the string vertex operators and boundary states. The origin of these phases is well-known: in toroidal compactifications the logarithmic branch cut of the bosonic Green function can sometime become visible and it is necessary to compensate for this by adding to the string vertices a phase known as cocycle (see for instance the paragraph “A technicality” in section 8.2 of [22]). All cocycle factors were ignored in [3], but this had no visible effects because, as we will show, these phases were not crucial in the particular examples considered there. However, in general, the partition function is unitary, in the open string channel, only when all phases have been taken into account. This comes as no surprise because the partition function of figure 1a contains a sum over all possible 3-string vertices and the effect of the cocycles becomes clearly visible since they yield relative phases between different terms. The presence of these cocycles has some consequences also on the precise formulation of the T-duality transformations. In fact these transformations should preserve both the spectrum and the interactions, including, in the latter case, the phases needed for ensuring the locality of the interactions. This does not happen with the naive version of the T-duality rules usually written, and we show that it is necessary to introduce some cocycle phases also in the T-duality transformation to recover a consistent picture.

Then we study the degeneration limit of the $g = 2$ partition function in the open string channel, see figure 1b, and derive an explicit form for the 3-twist field correlators on arbitrary tori. This computation, in presence of multiply wrapped D-branes, provides a concrete example showing that the cocycle phases are crucial for unitarity already at the level of the 2-loop partition function. Our result on the 3-twist field couplings generalizes previous expressions valid for configurations that are completely factorized in a product of 2-tori [12–16, 3], which requires that both the background geometry and the D-brane gauge field strengths are non-trivial only along the same orthogonal T^2 's and all “off-diagonal” entries are put to zero. Even in presence of commuting fluxes this is a very particular situation, where the so-called classical contribution to the 3-twist field correlators is written in terms of Jacobi Theta-functions. In general, higher genus Theta-functions appear and so the Yukawa couplings that arise in non-factorized D-brane models will have a more complicated structure and a richer moduli dependence than those derived so far for the T^2 . However, we still find the same separation between Kähler and complex structure moduli and only the latter ones enter in the expressions for the classical contribution (this in the magnetized setup, of course the roles are switched in the language of D-branes at angles).

The structure of the paper is the following. In section 2 we briefly review the main features of the string dynamics in toroidal compactifications. We pay particular care to the cocycle factors in the definition of the vertices and of the boundary states. We show that

these phases enter in a non-trivial way also in the T-duality transformation of the states with non-zero Kaluza-Klein or winding numbers. In section 3 we revisit the computation of the twisted partition function discussed in [3] and include the effect of the Wilson lines and the D-brane multiple wrappings. The only hypothesis we still take is to restrict ourselves to the case of commuting monodromy matrices (see eq. (3.1)). In section 4 we focus on the $g = 2$ case and study the unitarity properties of the result. By doing so we derive the general expression for the 3-point correlators between abelian twist fields. Finally some technicalities are left to the appendix.

2. Toroidal compactifications

2.1 Some properties of T^{2d}

A $2d$ -dimensional real torus T^{2d} is defined by a collection of $2d$ vectors a_M in the Euclidean space \mathbb{R}^{2d} : T^{2d} is simply \mathbb{R}^{2d} modulo the identification with integer shifts along the a_M 's

$$x \equiv x + 2\pi\sqrt{\alpha'} \sum_{M=1}^{2d} c^M a_M \quad , \quad \forall x \in \mathbb{R}^{2d} \quad \text{and} \quad c^M \in \mathbb{Z} . \quad (2.1)$$

One can choose a Cartesian reference frame and write the components of each vector a_M as a_M^a . Then the metric on T^{2d} is² $G_{MN} = \sum_{a=1}^{2d} a_M^a a_N^a$. The components of each a_M can be used to fill in the column vectors of a square matrix $E_M^a \equiv a_M^a$; then, by construction, E is a vielbein matrix satisfying

$$G = {}^t E E \quad , \quad 1 = {}^t E^{-1} G E^{-1} . \quad (2.2)$$

The inverse matrix E^{-1} instead has the dual vectors \hat{a}^M as rows: $\sum_{a=1}^{2d} \hat{a}_a^M a_N^a = \delta_N^M$. Of course any other matrix $E' = O E$ obtained by means of an orthogonal rotation O of E is a good vielbein matrix. We can also introduce complex coordinates and the complex vielbein \mathcal{E} , such that

$$\mathcal{E} = S E \quad , \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} , \quad (2.3)$$

where all the four blocks of S are proportional to the $d \times d$ identity matrix. Two sets of complex coordinates are inequivalent if they cannot be connected by a unitary transformation. Thus the $SO(2d)$ ambiguity in the definition of the E 's implies that on the same real torus there is a set of inequivalent complex structures which is parametrized by $SO(2d)/U(d)$. Notice that in the complex coordinates the flat metric \mathcal{G} is off-diagonal

$$\mathcal{G} = {}^t \mathcal{E}^{-1} G \mathcal{E}^{-1} = \bar{S} S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \quad (2.4)$$

In presence of a constant antisymmetric two-form, there is a particular complex structure that plays a special role. Of course in our applications this 2-form will be the gauge

²In our conventions the coordinates with the indices M, N, \dots are parallel to the lattice defining the torus; they have period $2\pi\sqrt{\alpha'}$ and are referred to as the integral basis.

invariant combination between the Kalb-Ramond B -field and F , the magnetic field on the D-Branes: $\mathcal{F} = B + F$, thus we will indicate this antisymmetric tensor with \mathcal{F} already in this section (notice the different convention with respect to [3], as here the factor of $2\pi\alpha'$ in front of F is reabsorbed in the definition of a dimensionless magnetic field). Its expression in a real cartesian basis is

$$\mathcal{F}_c = {}^tE^{-1}\mathcal{F}E^{-1} = EG^{-1}\mathcal{F}E^{-1} . \tag{2.5}$$

and is related by a similarity transformation to the combination $G^{-1}\mathcal{F}$ which will be the most relevant in the following sections. The antisymmetric matrix \mathcal{F}_c can be reduced to a block-diagonal form $\mathcal{F}_{\text{b.d.}}$ by means of an orthogonal rotation O_f (where the subscript is just to recall that this transformation depends in general on \mathcal{F})

$$\mathcal{F}_{\text{b.d.}} = \begin{pmatrix} 0 & \mathfrak{f}_d \\ -\mathfrak{f}_d & 0 \end{pmatrix} = {}^tE_f^{-1}\mathcal{F}E_f^{-1} = E_fG^{-1}\mathcal{F}E_f^{-1} , \tag{2.6}$$

where \mathfrak{f}_d is a $d \times d$ diagonal matrix with real entries \mathfrak{f}_{aa} . The vielbein matrix $E_f = O_fE$ transforms at the same time the metric G into the identity and \mathcal{F} into the block-diagonal matrix (2.6). Of course we can use the vielbein matrix E_f to introduce a particular set of complex coordinates ($\mathcal{E}_f = SE_f$) which diagonalizes $G^{-1}\mathcal{F}$

$$\mathcal{F}^{(d)} = \mathcal{E}_fG^{-1}\mathcal{F}\mathcal{E}_f^{-1} = \begin{pmatrix} -i\mathfrak{f}_d & 0 \\ 0 & i\mathfrak{f}_d \end{pmatrix} = \mathcal{G}{}^t\mathcal{E}_f^{-1}\mathcal{F}\mathcal{E}_f^{-1} , \tag{2.7}$$

where ${}^t\mathcal{E}_f^{-1}\mathcal{F}\mathcal{E}_f^{-1}$ is a block-diagonal matrix. From (2.7) it is easy to see that, in this complex basis, \mathcal{F} is a $(1,1)$ -form. In the following we will always use, as Cartesian basis, the one defined by the vielbeins E_f or \mathcal{E}_f , thus we will drop the subscripts without risk of ambiguities. Finally let us recall the definition of the complex structure as the mixed tensor

$$\mathcal{I} = idz \otimes \frac{\partial}{\partial z} - id\bar{z} \otimes \frac{\partial}{\partial \bar{z}} . \tag{2.8}$$

In the following sections we will need the expression of $\mathcal{I}_a{}^b$ in the integral basis, which is easily derived by using the complex vielbein \mathcal{E} :

$$\mathcal{I}_M{}^N = ({}^t\mathcal{E})_M{}^a \mathcal{I}_a{}^b ({}^t\mathcal{E}^{-1})_b{}^N . \tag{2.9}$$

2.2 Closed strings on T^{2d}

The coordinates of the closed strings propagating on T^{2d} with a constant background B -field can be written as a sum of left and right handed fields, and the world-sheet is described by a free CFT. Then we have the usual mode expansion

$$x_{\text{cl}}^M(z, \bar{z}) = \frac{X_{\text{cl}}^M(z) + \tilde{X}_{\text{cl}}^M(\bar{z})}{2}, \quad \text{where} \quad X_{\text{cl}}^M(z) = x^M - i\sqrt{2\alpha'}\alpha_0^M \ln z + i\sqrt{2\alpha'} \sum_{m \neq 0} \frac{\alpha_m^M z^{-m}}{m}, \tag{2.10}$$

and the complexified world-sheet coordinates are

$$z = e^{\tau+i\sigma} \quad \text{and} \quad \bar{z} = e^{\tau-i\sigma}, \quad (2.11)$$

τ and σ being the world-sheet time and spacial coordinates respectively. Upon canonical quantization, the commutation relations for the oscillators in eq. (2.10) are

$$[\alpha_m^M, \alpha_{-n}^N] = n\delta_{m,n}G^{MN}, \quad \text{and} \quad [x^M, \alpha_0^N] = i\sqrt{2\alpha'}G^{MN}, \quad (2.12)$$

and similarly in the right moving sector. The allowed winding and Kaluza-Klein modes are encoded in the Narain lattice which depends on the background fields G and B as follows

$$\alpha_0^M = \frac{G^{MN}}{\sqrt{2}} \left[\hat{n}_N + (G_{NN'} - B_{NN'}) \hat{m}^{N'} \right], \quad \tilde{\alpha}_0^M = \frac{G^{MN}}{\sqrt{2}} \left[\hat{n}_N - (G_{NN'} + B_{NN'}) \hat{m}^{N'} \right], \quad (2.13)$$

where \hat{n}_M and \hat{m}^M are respectively the Kaluza-Klein and winding numbers operators.

Let us now consider the interacting theory and pay particular attention to the phases necessary when the target space is compact. The basic building block is the tree-level coupling among three generic closed strings (Reggeon vertex). Since there is no preferred ordering of three points on a sphere, this vertex must be invariant under the action of the permutation group exchanging any of the punctures. The usual emission vertex valid in the uncompact space does not have this property when it is naively generalized to the T^{2d} case. This is a well known issue, related to the compactness of the target space, and it is a consequence of the logarithmic branch cut of the bosonic Green function. Similar problems arise also in the usual formalism of the vertex operators describing the emission of particular on-shell string states (see, for instance, section 8.2 in [22]). In order to eliminate this branch cut one has to add suitable cocycle factors to the usual expression of the vertex operators. Here we tackle this issue exactly in the same way by generalising these cocycle factors to the Reggeon vertex formalism.

In order to see that the usual 3-string vertex for closed strings is not invariant under the permutation of the external states when the target space is a torus, it is sufficient to focus on the zero-modes contribution. The explicit form of the full vertex V_3 can be found, for instance, in [3] and its zero-mode part is simply

$$\exp \left[\sum_{j>i=0}^2 \alpha_0^i \ln(z_j - z_i) G \alpha_0^j + \sum_{j>i=0}^2 \tilde{\alpha}_0^i \ln(\bar{z}_j - \bar{z}_i) G \tilde{\alpha}_0^j \right], \quad (2.14)$$

where the z_i ($i = 0, 1, 2$) are the positions of the three punctures on the sphere (which is represented as the compactified complex plane); the upper index identifies one of the three external states, and all space-times indices have been suppressed. The oscillator part of the 3-string vertex is invariant under the exchange of the strings $1 \leftrightarrow 2$, while the zero-mode contribution (2.14) gets a phase given by

$$\exp \left[-i\pi (\alpha_0^1 G \alpha_0^2 - \tilde{\alpha}_0^1 G \tilde{\alpha}_0^2) \right] = \exp \left[-i\pi (\hat{n}_1 \hat{m}_2 + \hat{m}_1 \hat{n}_2) \right], \quad (2.15)$$

where we have used (2.13). By using this result, we can build a new invariant vertex with a cocycle factor that compensates for the phase (2.15). A possible choice for this cocycle factor is

$$V_3^c = V_3 \exp \left[\frac{i\pi}{2} \left(\hat{n}_1 \hat{m}_2 - \hat{m}_1 \hat{n}_2 \right) \right]. \quad (2.16)$$

By using the conservation of the Kaluza-Klein and winding numbers of the emitted strings, the vertex (2.16) is now easily shown to be invariant under the full permutation group acting on the three punctures.

However, the cocycle factor added seems to break the vertex invariance under T-Duality transformations. For instance, there is a particular T-duality transformation that exchanges the Kaluza-Klein and winding operators. If its effect could be simply written as $\hat{n} \leftrightarrow \hat{m}$, as it is usually done, then we could have that $V_3^c \rightarrow -V_3^c$ for certain external states. In order to clarify this point, let us see the transformation properties of V_3^c under a generic T-duality transformation. These transformations can be encoded in a $O(2d, 2d, \mathbb{Z})$ matrix [23]

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with} \quad a^t b + b^t a = 0, \quad c^t d + d^t c = 0, \quad a^t d + b^t c = 1, \quad (2.17)$$

where a , b , c and d are $2d \times 2d$ matrices and their constraints follow from the group definition, namely

$$J = T J^t T, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.18)$$

Let us observe that, as $J = J^{-1}$, by inverting the relation above, one finds that also ${}^t T$ is an $O(2d, 2d, \mathbb{Z})$ matrix with similar constraints imposed on its entries. The duality acts as follows on the geometrical background

$$G' + B' = [a(G + B) + b] [c(G + B) + d]^{-1} \quad (2.19)$$

and on the string oscillators

$$\alpha_n = T_+ \alpha'_n, \quad \tilde{\alpha}_n = T_- \tilde{\alpha}'_n, \quad (2.20)$$

where

$$T_{\pm} = [c(\pm G + B) + d]^{-1}. \quad (2.21)$$

Notice that one can prove [23] that ${}^t T_{\pm} G T_{\pm} = G'$. Focusing now on the zero-modes, we can derive from (2.20) the action of the T-duality transformation on the winding and Kaluza-Klein operators

$$\hat{n} = {}^t d \hat{n}' + {}^t b \hat{m}', \quad \hat{m} = {}^t c \hat{n}' + {}^t a \hat{m}' \iff \hat{n}' = a \hat{n} + b \hat{m}, \quad \hat{m}' = c \hat{n} + d \hat{m} \quad (2.22)$$

Then the transformation law for the vertex (2.16) is

$$\begin{aligned} V_3^c &= V_3^{c'} e^{i\pi [{}^t \hat{n}'_1 d^t \hat{c} \hat{n}'_2 + {}^t \hat{m}'_1 b^t \hat{a} \hat{m}'_2 + {}^t \hat{n}'_1 d^t \hat{a} \hat{m}'_2 + {}^t \hat{m}'_1 b^t \hat{c} \hat{n}'_2 - {}^t \hat{n}'_1 \hat{m}'_2]} \\ &= V_3^{c'} \prod_{j=0}^2 \left\{ e^{i\pi [\sum_{M < N} ((\hat{n}'_j)_M (d^t c)^{MN} (\hat{n}'_j)_N + (\hat{m}'_j)^M (b^t a)_{MN} (\hat{m}'_j)^N) + {}^t \hat{m}'_j b^t \hat{c} \hat{n}'_j]} \right\}, \end{aligned} \quad (2.23)$$

where in the last line we used of the conservation of windings and momenta. Thus the vertex (2.16) is fully symmetric under permutations of the external states, but is not invariant under the T-Duality transformations (2.20). However, it is interesting to notice that the phase (which is actually just a sign) generated by the transformations (2.20) can always be written as a product of three terms each one depending only a single external state, as it is done in the second line of (2.23). This means that the invariance of the vertex (2.16) under T-Duality can be restored, provided that we introduce the appropriate cocycle also in the T-duality transformations, as a generalisation of the standard rules discussed above and in [23]. In fact it is sufficient to postulate that the closed string states transform according to (2.22) *and* also acquire the same phase in the curly brackets of (2.23)

$$|n, m\rangle \rightarrow e^{i\pi[\sum_{M<N}((\hat{n}')_M(d^t c)^{MN}(\hat{n})_N+(\hat{m}')^M(b^t a)_{MN}(\hat{m}')^N)+^t \hat{m}' b^t c \hat{n}']} |n', m'\rangle . \quad (2.24)$$

We will see in the following section that this is indeed the case when considering boundary states describing magnetised D-Branes as dual for instance to configurations of purely Dirichlet or intersecting D-Branes.

It is not difficult to generalize the analysis above to the case of a Reggeon vertex describing the interaction of many closed strings. This can be obtained just by gluing together the 3-point vertices (2.16) and the result is

$$V_{N+1}^c = V_{N+1} \exp \left[\frac{\pi i}{2} \sum_{j>i=1}^N \left(\hat{n}_i \hat{m}_j - \hat{m}_i \hat{n}_j \right) \right] . \quad (2.25)$$

Here V_{N+1} is the standard vertex, where the cocycles are ignored and, as before, the indices $i, j = 0, 1, \dots, N$ label the $N + 1$ external states.

2.3 The Boundary State for a wrapped magnetized D-Brane

In this section we study, from the closed string point of view, the space-filling magnetised D-Branes with generic wrapping numbers on the torus cycles (see [24, 25] for previous works on this subject). In the closed string sector D-Branes are described by boundary states $|B_F\rangle$ that enforce an identification between the left and right moving modes (for a review of the boundary state formalism in the Ramond Neveu-Schwarz formalism see [26, 27]). For magnetized D-branes we have

$$[(G + \mathcal{F})\alpha_n + (G - \mathcal{F})\tilde{\alpha}_{-n}]|B_F\rangle = 0, \quad \forall n \in \mathbb{Z}, \quad (2.26)$$

where we used the gauge invariant combination of the Kalb-Ramond B -field and the field strength F on the D-brane: $\mathcal{F} = B + F$. In the integral basis, the magnetic field is quantized as a consequence of the compactness of the torus,

$$F_{MN} = \frac{p_{MN}}{w_M w_N}, \quad (2.27)$$

p_{MN} being an integer matrix and w_M being the wrapping numbers of the D-Brane along the cycles of the torus. Clearly eq. (2.26), being a set of linear constraints, fixes the form

of the boundary state up to an overall factor that can also depend on the Kaluza-Klein and winding operators. We will now show that the boundary state for a magnetised D-Brane does indeed contain non-trivial phases that depend on the winding numbers and on the field F ; moreover it turns out that this phase is strictly related to the phase that closed string states acquire under T-Duality. Following [28], we consider the gauge field contribution to the action in the string path-integral as an interaction term that acts on the standard boundary state for an unmagnetised D-Brane wrapping the T^{2d} . We want to derive the dependence of $|B_F\rangle$ on the magnetic field by applying the usual path ordered Wilson loop operator $P[\exp(\frac{i}{2\pi\alpha'} \oint Adx)]$ to $|B_{F=0}\rangle$. The computation was performed in [28] in a Minkowski flat target space, while the novelty of the present calculation³ is the compactness of the torus wrapped by the D-Branes. As the non-zero modes contribution to the boundary state is not affected by the shape of the target space, we will mostly focus on the zero-modes. Of course, our aim is to determine all the F -dependent terms that cannot be fixed from (2.26). In order to do so we have to pay some care to the definition of the Wilson loop operator.

2.3.1 Gauge bundles and gauge invariant Wilson loop

Any gauge potential for (2.27) will involve a linear, and thus non-periodic, function. Let us start from the simpler case $w_M = 1, \forall M$. As usual, we need to compensate for the non-periodicity of A by introducing a set of gauge transformations U_N . Each U_N encodes the gluing conditions for the gauge potential between two copies of the torus that are adjacent along the N -th direction in the covering space. So the gauge potential living on a D-Branes world volume wrapping a cycle of the compactification torus must satisfy

$$A_M(x + 2\pi\sqrt{\alpha'}a_N) = U_N(x) (2\pi\alpha' i\partial_M + A_M(x)) U_N^\dagger(x), \quad (2.28)$$

where a_N denotes the N -th cycle of the torus. In the case under analysis, all gauge transformations U_N belong to $U(1)$ and so the formula above can be further simplified. We choose not to do it, so as to keep the equations (2.28)–(2.31) valid also for the non-abelian generalization that we will need once we reintroduce multiple wrappings. The background gauge field (gauge bundle) is properly defined by eq. (2.27) together with the set of U_N 's. In order to have a consistent bundle, the gluing matrices must satisfy the overlap condition

$$U_N^\dagger(x) U_M^\dagger(x + 2\pi\sqrt{\alpha'}a_N) U_N(x + 2\pi\sqrt{\alpha'}a_M) U_M(x) = 1. \quad (2.29)$$

All fields charged under the gauge potential have to obey periodicity conditions similar to (2.28). For instance, fields transforming in the fundamental (Φ) or in the adjoint (Ψ) representation must satisfy

$$\Phi(x + 2\pi\sqrt{\alpha'}a_N) = U_N(x)\Phi(x), \quad \Psi(x + 2\pi\sqrt{\alpha'}a_N) = U_N(x)\Psi(x)U_N^\dagger(x). \quad (2.30)$$

As a consequence of eq. (2.30), under a generic gauge transformation $\gamma(x)$, the gluing matrices U_N transform in the following fashion

$$U_N(x) \rightarrow \gamma(x + 2\pi\sqrt{\alpha'}a_N) U_N(x) \gamma^\dagger(x). \quad (2.31)$$

³An analogous calculation is performed in [29]; we thank I. Pesando for letting us know their results before publication.

Notice that there are no restrictions on $\gamma(x)$ and in particular it does not have to be periodic. So the form of the U_N 's can change under a gauge transformation.

We now focus on the definition of the Wilson loop operator we need for the computation of the magnetized boundary state. Let us consider a path c connecting two points that are separated by the lattice vector $\sum_{L=1}^{2d} m_L a_L$, $m_L \in \mathbb{Z}$ (which means that they are identified on the torus). In order to be consistent with our convention for \mathcal{F} , this path will start from $x + 2\pi\sqrt{\alpha'}$ $\sum_{L=1}^{2d} m_L a_L$ and end in x . Then it is clear that the naive path ordered Wilson loop operator is not gauge invariant,⁴ but transforms as

$$P[e^{i \int_c A dx}] \rightarrow \gamma(x) P[e^{i \int_c A dx}] \gamma^\dagger \left(x + 2\pi\sqrt{\alpha'} \sum_{L=1}^{2d} m_L a_L \right). \quad (2.32)$$

We can recover a gauge invariant object if we multiply the Wilson loop (2.32) by a sequence of U 's which forms a discretized version of the path c . By using (2.29) we can choose to collect together all shifts along the direction $K = 1$, then those along $K = 2$ and so on. In formulae we have

$$\left[\prod_{K=1}^{2d} \prod_{m=0}^{m_K-1} U_K \left(x + 2\pi\sqrt{\alpha'} \sum_{L=1}^{K-1} m_L a_L + 2\pi\sqrt{\alpha'} m a_K \right) \right] P[e^{\frac{i}{2\pi\alpha'} \int_c A dx}], \quad (2.33)$$

where only the values $m_K \geq 1$ are relevant (if we have $m_K = 0$ for certain K , then the corresponding U_K does not appear in the product). By using (2.31) and (2.32) (and (2.28)), one can check that (2.33) is invariant under an arbitrary $U(1)$ gauge transformation $\gamma(x)$ (and does not depend on the initial point x of the path c).

2.3.2 Wrapped D-Branes as non abelian gauge bundles

A D-brane with multiple wrappings ($w_M > 1$ for some M) is better described [30, 31] in terms of a non-trivial gauge bundle on the torus T^{2d} [32]. In the D-brane language this amounts to considering a set of $w = \prod_{M=1}^{2d} w_M$ coincident D-branes with the same gauge field strength (2.27), but with non-trivial transition matrices U_M . The non-abelian character of the configuration is encoded in the U_M 's that, as consequence of (2.30), glue together the various D-branes in a single wrapped object. In absence of magnetic fields, this can be easily seen by choosing as U_M the following $w_M \times w_M$ transition matrix

$$U_M = P_{w_M \times w_M} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (2.34)$$

Notice that, when $F = 0$, a D-Brane wrapped w_M times along the M -th cycle of the torus can be smoothly deformed, at the classical level, into w_M coincident branes [33, 34]. This means that there is a family of flat gauge bundles interpolating between (2.34) and $U_M = 1$.

⁴and even depends on the initial point x of the path c .

In order to introduce the effect of the magnetic field (2.27) on the D-Brane, it turns out to be convenient to choose a fundamental cell of the torus lattice where this field is in a block-diagonal form. This can be always done [35] (see also the appendix for a more pedestrian proof), so from now on we take

$$F = \begin{pmatrix} 0 & \frac{p_1}{W_1} & 0 & 0 & \cdots \\ -\frac{p_1}{W_1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{p_2}{W_2} & \cdots \\ 0 & 0 & -\frac{p_2}{W_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.35)$$

where $p_\alpha \in \mathbb{Z}$ and $W_\alpha \in \mathbb{Z} - \{0\}$, $\forall \alpha = 1, \dots, d$. Notice that even if the field F now describes a direct product of d T^2 's inside the T^{2d} , the compactification is in general non factorizable as a consequence of the form of the metric. The p_α 's can be interpreted as the Chern classes of the magnetic fields while the W_α 's are the products of the couples of wrapping numbers on each of the T^2 's inside T^{2d} .

Two comments are in order now. First, if p_α and W_α are not co-prime, the configuration can again be smoothly deformed, at the classical level, to a new configuration with co-prime p'_α and W'_α and $p'_\alpha/W'_\alpha = p_\alpha/W_\alpha$. Second it is not necessary to specify all the wrappings of the brane along each cycle of the torus. In fact it is possible to show that configurations of magnetized D-Branes with the same product of wrappings along the pairs of cycles of the T^2 's inside the T^{2d} defined by the form of the field (2.35) are equivalent. Indeed the transition matrices defining the gauge bundle of a brane wrapped along a T^2 are related by a gauge transformation if they describe branes with the same Chern class p and the same product of the wrappings W . Thus we can choose to wrap the branes W_α times along the even directions $x_M \equiv x_{2\alpha}$ only. We will also make the following gauge choice for the gauge potential (2.35)

$$A_{M \equiv 2\alpha}(x) = \frac{p_\alpha}{W_\alpha} x_{2\alpha-1} + 2\pi\sqrt{\alpha'} C_{2\alpha-1} \quad \text{and} \quad A_{M \equiv 2\alpha-1}(x) = 2\pi\sqrt{\alpha'} C_{2\alpha}, \quad \forall \alpha = 1, \dots, d, \quad (2.36)$$

where we have introduced non-zero Wilson lines $2\pi\sqrt{\alpha'} C_M$, with C_M adimensional. In this way the non abelian gauge bundle describing the magnetized wrapped D-Brane is characterized by

$$U_{2\alpha}(x) = 1_{W_1 \times W_1} \otimes \dots \otimes P_{W_\alpha \times W_\alpha} \otimes 1_{W_{\alpha+1} \times W_{\alpha+1}} \otimes \dots \otimes 1_{W_d \times W_d} \quad (2.37)$$

$$U_{2\alpha-1}(x) = 1_{W_1 \times W_1} \otimes \dots \otimes (Q_{W_\alpha \times W_\alpha})^{p_\alpha} \otimes 1_{W_{\alpha+1} \times W_{\alpha+1}} \otimes \dots \otimes 1_{W_d \times W_d} e^{\frac{i}{\sqrt{\alpha'}} \frac{p_\alpha}{W_\alpha} x_{2\alpha}},$$

where $Q_{W_\alpha \times W_\alpha} = \text{diag} \{1, e^{2\pi i/W_\alpha}, \dots, e^{2\pi i(W_\alpha-1)/W_\alpha}\}$. Notice that the form of the transition matrices U_M is not affected by the presence of the Wilson-lines as for them the relation (2.28) is trivially satisfied by the identity gluing matrix.

The generalization of (2.33) to this non-abelian setup is straightforward and the operator we need to use to derive the magnetized boundary state from the unmagnetized one

reads

$$\mathcal{O}_A = \text{Tr} \left\{ \left[\prod_{K=1}^{2d} \prod_{m=0}^{\hat{m}_K-1} U_K \left(x_{\text{cl}} + \sum_{L=1}^{K-1} 2\pi\sqrt{\alpha'} \hat{m}_{L A L} + 2\pi\sqrt{\alpha'} m a_K \right) \right] P \left[e^{\frac{i}{2\pi\alpha'} \int_c A dx_{\text{cl}}} \right] \right\}, \quad (2.38)$$

where the x_{cl}^M 's are the usual string coordinates (2.10) and the \hat{m}_M 's are the operators that read the winding numbers of the closed string states. In this non-abelian generalization we have to put $U(x)$ at the right hand of the sequence and then follow the order determined by the path c .

2.3.3 Computation of the Boundary State

We can now compute the action of \mathcal{O}_A on the unmagnetized boundary state $|B_{F,C}\rangle = \mathcal{O}_A |B_{A=0}\rangle$. The unmagnetized boundary state for a wrapped D-Brane is found from the one of an unwrapped brane (see [27] and references therein) by applying the same operator (2.38) with the choice

$$A_M = 0 \quad \text{and} \quad U_K = 1_{w_1 \times w_1} \otimes \dots \otimes P_{w_K \times w_K} \otimes 1_{w_{K+1} \times w_{K+1}} \otimes \dots \otimes 1_{w_{2d} \times w_{2d}}, \quad (2.39)$$

with P defined as in eq. (2.34). In this case the trace in (2.38) reads

$$\text{Tr} \left[\prod_{K=1}^{2d} U_K^{\hat{m}_K} \right] \quad (2.40)$$

and it is different from zero only when the windings of the emitted closed strings are integer multiples of the wrappings of the D-Brane on each cycle of the torus. Hence only these states couple to the wrapped D-brane, as expected, and we have

$$|B_{A=0}\rangle = \sqrt{\text{Det}(G+B)} \sum_{m^M \in Z} \prod_{n=1}^{\infty} e^{-\frac{1}{n} \tilde{\alpha}_n^\dagger G R_0 \alpha_n^\dagger} |0; w_M m^M\rangle, \quad (2.41)$$

where there is no sum understood over the repeated index M ; $R_0 = (G-B)^{-1}(G+B)$ is the identification matrix between left and right moving oscillators and depends on the geometric background of the torus; finally the ket $|0; w_M m^M\rangle$ represents the closed string vacuum state with zero Kaluza-Klein momenta and winding numbers equal to $w_M m^M$ for $M = 1, 2, \dots, 2d$.

It is easy to begin by turning on only the Wilson lines and to keep vanishing magnetic fields on the D-brane world volume. We have just to isolate the Wilson line contribution to \mathcal{O}_A in (2.38) when acting on $|B_{A=0}\rangle$, namely

$$|B_C\rangle = e^{\frac{i}{\sqrt{\alpha'}} \int_c C \cdot dx} |B_{A=0}\rangle = e^{2\pi i C \cdot \hat{m}} |B_{A=0}\rangle. \quad (2.42)$$

The explicit evaluation of the contributions of the magnetic fields F is longer. It can be split into the zero and non-zero mode part of the string coordinate $x_{\text{cl}}(z, \bar{z})$ defined in (2.10). Let us focus on the zero-mode part of the computation, since the non-zero mode contribution has just the effect of replacing B with \mathcal{F} in (2.41), see [28]. A first result is

that the magnetized boundary state couples only with the closed strings whose windings in the α -th T^2 (as defined by the form of the magnetic field in (2.35)) are integer multiples of W_α . This is again a consequence of the trace in (2.38) where the transition matrices are defined as in eq. (2.37). It is convenient to define the $2d \times 2d$ matrix

$$w = \begin{pmatrix} W_1 & 0 & \cdots \\ 0 & W_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \otimes 1_{2 \times 2} \quad (2.43)$$

and use wm to indicate the $2d$ column vector containing the windings of the closed strings emitted by the magnetized D-Brane. We can also see how the action of \mathcal{O}_A yields the relation between windings and Kaluza-Klein numbers

$$\hat{n} = -F\hat{m}, \quad (2.44)$$

which is usually derived from the identification imposed by eq. (2.26) on the closed string zero-modes. For this, it is sufficient to focus on the zero-modes contribution linear both in the position operator $\hat{q}^M = (x^M + \tilde{x}^M)/2$ and in the oscillators α_0 or $\tilde{\alpha}_0$. Using the form of F in eq. (2.35) with the gauge choice (2.36) and the transition matrices (2.37), we can evaluate this contribution as follows

$$\begin{aligned} |B_F\rangle &\sim e^{\frac{i}{\sqrt{\alpha'}} \sum_{\alpha=1}^d \frac{p_\alpha}{W_\alpha} \hat{m}_{2\alpha-1} \hat{q}_{2\alpha}} e^{-i \frac{\sqrt{\alpha'}}{2\sqrt{2}} \sum_{\alpha=1}^d \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'} \frac{p_\alpha}{W_\alpha} (x_{2\alpha-1} + \tilde{x}_{2\alpha-1})(\alpha_0^{2\alpha} - \tilde{\alpha}_0^{2\alpha})} |0; wm\rangle \\ &= e^{i \sum_{\alpha=1}^d \frac{p_\alpha}{W_\alpha} \hat{m}_{2\alpha-1} \frac{\hat{q}_{2\alpha}}{\sqrt{\alpha'}}} e^{-i \sum_{\alpha=1}^d \frac{p_\alpha}{W_\alpha} \frac{\hat{q}_{2\alpha-1}}{\sqrt{\alpha'}} \hat{m}_{2\alpha}} |0; wm\rangle = |-Fwm, wm\rangle. \end{aligned} \quad (2.45)$$

Notice that at this stage we can forget about the path-ordering in the Wilson operator \mathcal{O}_A and explicitly perform the integration in the first line of the previous equation, as the zero-mode contributions of the string fields entering the Wilson loop in \mathcal{O}_A commute with each other at different values of σ . Finally let us consider the terms quadratic in the zero-modes α_0 and $\tilde{\alpha}_0$ that follow from the standard Wilson loop exponential in eq. (2.38),

$$\begin{aligned} e^{-\frac{i}{2} \sum_{\alpha=1}^d \int_0^{2\pi} d\sigma \frac{1}{2\pi} \frac{p_\alpha}{W_\alpha} (\alpha_0^{2\alpha-1} \alpha_0^{2\alpha} - \tilde{\alpha}_0^{2\alpha-1} \alpha_0^{2\alpha} - \alpha_0^{2\alpha-1} \tilde{\alpha}_0^{2\alpha} + \tilde{\alpha}_0^{2\alpha-1} \tilde{\alpha}_0^{2\alpha}) \sigma} |-Fwm; wm\rangle \\ = e^{-i\pi \sum_{\alpha=1}^d W_\alpha p_\alpha m_{2\alpha-1} \hat{m}_{2\alpha}} |-Fwm; wm\rangle = e^{-i\pi \sum_{M<N} \hat{m}^M F_{MN} \hat{m}^N} |-Fwm; wm\rangle. \end{aligned} \quad (2.46)$$

The phase in (2.46) could not have been deduced just by looking at the constraints (2.26). So the expression for the boundary state describing a magnetised D-Brane is

$$\begin{aligned} |B_{F,C}\rangle &= \sqrt{\text{Det}(G + \mathcal{F})} \sum_{m \in \mathbb{Z}^{2d}} e^{-i\pi \sum_{M<N} \hat{m}^M F_{MN} \hat{m}^N} e^{2\pi i C \hat{m}} \\ &\times \left[\prod_{n=1}^{\infty} e^{-\frac{1}{n} \tilde{\alpha}_n^\dagger G R \alpha_n^\dagger} \right] |-Fwm, wm\rangle, \end{aligned} \quad (2.47)$$

where the identification matrix R is

$$R = (G - \mathcal{F})^{-1} (G + \mathcal{F}). \quad (2.48)$$

Even if the phase in eq. (2.47) has been calculated for a block diagonal F , in the appendix we will prove that eq. (2.47) holds for a generic F (see eq. (2.27)), a part for possible half integer shifts in the Wilson line.

Let us now analyze how eq. (2.47) transforms under an $O(2d, 2d, \mathbb{Z})$ transformation (2.17). Generically, after the T-duality, we have a new magnetized D-brane with

$$F' = (aF - b)(-cF + d)^{-1}, \quad R' = T_-^{-1}RT_+, \quad (2.49)$$

as it can be seen by using the relations in (2.22); moreover, since F is an antisymmetric matrix, also F' is antisymmetric, thanks to constraints in (2.17). If the combination $(-cF + d)$ in (2.49) is not invertible, then the transformed D-brane will have some direction with Dirichlet boundary conditions. For instance, we can check that any magnetized brane can be easily related to a lower dimensional D-Brane at angles via T-Duality. First let us put the magnetic field in the block-diagonal form, as in eq. (2.35). Then, we T-dualize the even direction of each of the T^2 's defined by F inside the T^{2d} , that is, we choose the following $O(2d, 2d, \mathbb{Z})$ matrix

$$a = d = 1_{d \times d} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = c = 1_{d \times d} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.50)$$

By using the action of the duality transformations of the Kaluza-Klein and winding operators, as in eq. (2.22), the phase (2.46) can be rewritten as

$$e^{-i\pi \sum_{\alpha=1}^d \frac{p_{\alpha}}{W_{\alpha}} (W_{\alpha}^2) m_{2\alpha-1} m_{2\alpha}} = e^{-i\pi \sum_{\alpha=1}^d \hat{m}'_{2\alpha} \hat{n}'_{2\alpha}}. \quad (2.51)$$

Notice that this phase exactly compensates for the one that the closed string states in the ket of (2.47) acquire under T-Duality, as stated in eq. (2.24). Thus we see that the boundary states for purely geometrical configurations of D-Branes (like brane intersecting at angles) do not contain any non trivial phase depending on the emitted closed strings zero-modes, as expected. By reinstating the g_s dependence and using the results of [36], it is possible to show that also the prefactor $\sqrt{\text{Det}(G + \mathcal{F})}$ transforms into the one expected for the boundary state of a D-Brane at angle.

It is also possible to transform any magnetized D-brane into a D-brane with Dirichlet boundary conditions along all the coordinates of the torus. In this case, the matrices c and d defining the T-duality are related to the magnetic field $F = c^{-1}d$. When F is block-diagonal (2.35), one can choose $c = w$ given by (2.43) and easily build integer matrices a and b satisfying eqs. (2.17) using the fact that the wrappings W_{α} and the Chern numbers p_{α} are coprime. Exactly as in the previous example, the phase of the magnetized boundary state cancels against the phase generated by the T-duality transformation (2.24). Thus one recovers the standard form of a Dirichlet boundary state, where the identification matrix is simply $R = -1$.

Finally the transformations on the closed strings zero-modes (2.24) show explicitly that the Wilson lines in (2.47) are related to the positions of the D-Brane if the dualized directions have Dirichlet boundary condition. For the case of the D-branes at angles, in each of the two-dimensional tori inside the T^{2d} , half of the components of the Wilson lines becomes positions and the remaining ones are still interpreted as residual Wilson lines on the dualized brane. When the D-brane is transformed into a point in the compact space, then all components of the Wilson lines are geometrized into positions of the dual D-brane.

3. The twisted partition function

In this section we will compute the bosonic contribution to the open string twisted partition function for multiply wrapped D-branes. The planar partition function $Z_g(F)$ is obtained by starting from the tree-level vertex (2.25) and sewing the external legs with boundary states describing magnetized D-Branes with Wilson lines turned on (2.47). From the world-sheet point of view, this means that we start with a sphere and cut out $g + 1$ boundaries representing the magnetised D-branes. Thus we are dealing with a Riemann surface of genus zero, with $g + 1$ borders and no crosscaps; in the open string channel it corresponds to a g -loop diagram.

We start from the result obtained in [3] for D-branes without multiple wrappings and where all cocycle phases were ignored.⁵ In the open string channel, one of the D-branes (whose identification matrix is indicated with R_0) is singled out as the external border of the diagram; thus it is natural to introduce the monodromy matrices $\mathcal{S}_\mu \equiv R_0^{-1}R_\mu$, with $\mu = 1, \dots, g$, whose eigenvalues are $e^{\pm 2\pi i \epsilon_\mu^\alpha}$ ($\alpha = 1, 2, \dots, d$). The only assumption we will make on the monodromy matrices \mathcal{S}_μ is that they commute with each other, namely that

$$[\mathcal{S}_\mu, \mathcal{S}_\nu] = 0 . \tag{3.1}$$

Notice that this does not imply that the identification matrices R_i with $i = 0, \dots, g$ also commute with each other. Of course the converse holds and (3.1) is implied by the requirement that $[R_i, R_j] = 0$. By following the classification of [19] this more restrictive constraint is related to configurations with parallel magnetic fluxes. However, while eq. (3.1) is invariant under the T-Duality, the constraint among the identification matrices R is not, as they do not transform by a similarity transformation (see eq. (2.49)). Thus we will consider the slightly more general class of configurations satisfying (3.1). In this case, it is convenient to perform a T-Duality and transform the zero-th D-Brane into a purely Dirichlet D-Brane, i.e. with $R_0 = -1$. As discussed in the previous section, this can be done with a T-duality having $c^{-1}d = F_0$. With this choice we have $\mathcal{S}_\mu = -R_\mu$ and the commutator above can be rewritten as

$$[G^{-1}\mathcal{F}_\mu, G^{-1}\mathcal{F}_\nu] = 0 , \tag{3.2}$$

with $\mu = 1, \dots, g$ only. This implies that there exists a complex basis in which all the $G^{-1}\mathcal{F}_\mu$ are diagonal as in eq. (2.7). Moreover it implies:

$$[G^{-1}(F_{\hat{\mu}} - F_g), G^{-1}(F_{\hat{\nu}} - F_g)] = 0, \quad \forall \hat{\mu}, \hat{\nu} = 1, \dots, g - 1 ; \tag{3.3}$$

as a consequence, for a generic G one can deduce that a fundamental cell of the lattice torus exists where all the field differences $(F_{\hat{\mu}} - F_g)$ are simultaneously block diagonal. At any time, we can use again the T-duality rules discussed in the previous section and go back to the original system with all magnetized D-branes.

⁵We follow as much as possible the conventions of that paper.

We start from eq. (3.29) of [3] describing the g -loop partition function in the open string channel

$$Z_g(F) \sim \left[\prod_{\mu=1}^g \sqrt{\text{Det}(1 - G^{-1}\mathcal{F}_\mu)} \right] \int [dZ]_g \mathcal{A}^{(0)} \prod_{\alpha=1}^d \left[e^{-i\pi \bar{\epsilon}^\alpha \cdot \tau \cdot \bar{\epsilon}^\alpha} \frac{\det \tau}{\det T_{\bar{\epsilon}^\alpha}} \mathcal{R}_g(\bar{\epsilon}^\alpha \cdot \tau) \right], \quad (3.4)$$

where $[dZ]_g$ is the untwisted ($F_i = 0$) result and $\bar{\epsilon}^\alpha$ collects in a vector of length g all the twists ϵ^α ; τ and $T_{\bar{\epsilon}^\alpha}$ are the standard and the twisted period matrix respectively. $\mathcal{A}^{(0)}$ is the classical contribution to the partition function calculated in [3] in absence of the cocycle phases and setting all of the D-branes wrappings to one:

$$\mathcal{A}^{(0)} = \sum \Delta \exp \left\{ \pi i \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} \alpha_0^{\hat{\mu}} G \mathcal{S}_{\hat{\mu}}^{-1/2} \mathbf{D}_{\hat{\mu}\hat{\nu}} \mathcal{S}_{\hat{\nu}}^{1/2} \alpha_0^{\hat{\nu}} \right\}, \quad (3.5)$$

where $\mathbf{D}_{\hat{\mu}\hat{\nu}}$ is a space-time matrix determined by the \mathcal{S}_μ 's and the sum is over all the winding numbers that satisfy the Kronecker's deltas representing the identification (2.44) for each boundary state and the Kaluza-Klein and winding conservations (recall that closed strings emitted by the D-Brane with $R_0 = -1$ are characterized by unconstrained Kaluza-Klein momenta and no winding numbers):

$$\Delta = \left[\prod_{\mu=1}^g \delta(\hat{n}_\mu + F_\mu \hat{m}_\mu) \right] \delta \left(\sum_{i=0}^g \hat{n}_i \right) \delta \left(\sum_{\mu=1}^g \hat{m}_\mu \right). \quad (3.6)$$

All the ϵ -dependent ingredients, including the function \mathcal{R}_g and the matrix \mathbf{D} , are defined in [3] and, of course, depend on the moduli of the Riemann surface. We do not need the precise form of all these ingredients, but only some properties that we will recall later in this section. Notice that the classical contribution (3.5) is nontrivial only for $g \geq 2$: for the annulus we have $\mathcal{A}^{(0)} = 1$ and eq. (3.4) reduces to the partition function in the uncompact space [37].

In this section we complete (3.5) to include also the effects of the cocycle phases, multiple wrappings and open string moduli (Wilson lines and/or D-brane positions). The classical contribution is the only part of the partition function that is affected by this generalization, as it is clear from the form of the interaction vertex (2.25) and of the magnetized D-Branes boundary states (2.47). Basically we need to include in the sewing procedure the cocycle factor in (2.25) and the phases (2.47). Of course by following this approach we are effectively working in the closed string description. However the new contribution is independent of the world-sheet moduli and thus can be directly included in eq. (3.4) which is written in the open string channel. It is then clear that, in the expression for \mathcal{A} , we will have the same exponential of eq. (3.5) multiplied by some additional factors related to cocycles. So let us consider these new contributions: by using (2.25), (2.44)

and (2.47), one can see that all the phases from the cocycles and the Wilson lines yield

$$\begin{aligned} \exp \left\{ 2\pi i \left[\sum_{\mu=1}^g C_{\mu} \hat{m}_{\mu} + Y_0 \hat{n}_0 \right] \right\} \times \exp \left\{ \pi i \sum_{\mu=1}^g \sum_{M < N} \hat{m}_{\mu}^M (F_{\mu})_{MN} \hat{m}_{\mu}^N \right\} \\ \times \exp \left\{ \frac{\pi i}{2} \sum_{\nu > \mu=1}^g (\hat{m}_{\mu} F_{\mu} \hat{m}_{\nu} + \hat{m}_{\mu} F_{\nu} \hat{m}_{\nu}) \right\}. \end{aligned} \quad (3.7)$$

where Y_0 encodes the position of the zero-th D-Brane which is point-like along the torus directions. By using (3.6), we can eliminate \hat{m}_g from these sums. Then it is easy to see that we can rewrite the dependence on the open string moduli as follows:

$$\exp \left\{ 2\pi i \sum_{\hat{\mu}=1}^{g-1} [C_{\hat{\mu}} - C_g + Y_0 (F_{\hat{\mu}} - F_g)] \hat{m}_{\hat{\mu}} \right\} \equiv e^{2\pi i \sum_{\hat{\mu}=1}^{g-1} t_{\rho_{\hat{\mu}}} \hat{m}_{\hat{\mu}}}. \quad (3.8)$$

The second exponential in (3.7) comes from the boundary states; using the conservation of the winding numbers, the exponent can be rewritten as follows:

$$\sum_{\mu=1}^g \sum_{M < N} \hat{m}_{\mu}^M (F_{\mu})_{MN} \hat{m}_{\mu}^N = \sum_{\hat{\mu}=1}^{g-1} \sum_{M < N} \hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}})_{MN} \hat{m}_{\hat{\mu}}^N - \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} \sum_{M < N} \hat{m}_{\hat{\mu}}^M (F_g)_{MN} \hat{m}_{\hat{\nu}}^N. \quad (3.9)$$

By combining this contribution with the last exponent in eq. (3.7) and using (3.6) we get

$$\exp \left[-\pi i \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} \sum_{M < N} \hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}_{\hat{\nu}}^N \right], \quad (3.10)$$

where

$$F_{\hat{\mu}\hat{\nu}} = F_{\hat{\nu}\hat{\mu}} = F_{\hat{\nu}} - F_g, \quad \text{for } \hat{\nu} \geq \hat{\mu}. \quad (3.11)$$

Observe that in order to obtain this final form we have changed the sign of the combination $\sum_{M < N} \hat{m}_{\mu}^M (F_{\mu})_{MN} \hat{m}_{\mu}^N$ as each term of the sum is integer (see the appendix).

Thus the total contribution from the various phase factors is just the product of (3.8) and (3.10). This expression has no dependence on the metric of the torus and, in particular, eq. (3.10) provides just some relative signs between contributions related to different values of \hat{m} . Moreover it depends only on the differences $(F_{\hat{\nu}} - F_g)$; therefore, thanks to eq. (3.3), we can always consider $F_{\hat{\mu}\hat{\nu}}$ block diagonal.

Let us now reconsider the exponential in (3.5). By using (2.13), (2.44), (3.2) and the various conservation laws we can rewrite it as follows

$$\exp \left\{ \frac{\pi i}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} t_{\hat{m}_{\hat{\mu}}} G \left[1 - (G^{-1} \mathcal{F}_{\hat{\mu}})^2 \right]^{\frac{1}{2}} \mathbf{D}_{\hat{\mu}\hat{\nu}}(\mathcal{S}) \left[1 - (G^{-1} \mathcal{F}_{\hat{\nu}})^2 \right]^{\frac{1}{2}} \hat{m}_{\hat{\nu}} \right\}, \quad (3.12)$$

where we explicitly remind that \mathbf{D} is a function of the space-time matrices \mathcal{S} . It is convenient to rewrite (3.12) in the complex basis defined by the vielbein \mathcal{E} , where the $G^{-1} \mathcal{F}_i$'s

are diagonal and denoted by $\mathcal{F}_i^{(d)}$ as in eq. (2.7):

$$\exp \left\{ \frac{\pi i}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} {}^t \hat{m}_{\hat{\mu}} \left[{}^t \mathcal{E} \mathcal{G} \sqrt{\left(1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2\right) \left(1 - (\mathcal{F}_{\hat{\nu}}^{(d)})^2\right)} \begin{pmatrix} \mathcal{D}_{\hat{\mu}\hat{\nu}}(\epsilon) & 0 \\ 0 & \mathcal{D}_{\hat{\mu}\hat{\nu}}(-\epsilon) \end{pmatrix} \mathcal{E} \right] \hat{m}_{\hat{\nu}} \right\}, \quad (3.13)$$

where now each $\mathcal{D}_{\hat{\mu}\hat{\nu}}$ is $d \times d$ diagonal matrix that depends on the eigenvalues of the \mathcal{S}_{μ} 's.

The square parenthesis in (3.13) is contracted with a symmetric combination of \hat{m} , so we can symmetrize it. Then, by using (2.4), one can easily check that (3.13) is equal to

$$\exp \left\{ \frac{\pi i}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} {}^t \hat{m}_{\hat{\mu}} \left[{}^t \mathcal{E} \mathcal{G} \sqrt{\left(1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2\right) \left(1 - (\mathcal{F}_{\hat{\nu}}^{(d)})^2\right)} \begin{pmatrix} \hat{\tau}_{\hat{\mu}\hat{\nu}} & 0 \\ 0 & \hat{\tau}_{\hat{\nu}\hat{\mu}} \end{pmatrix} \mathcal{E} \right] \hat{m}_{\hat{\nu}} \right\}, \quad (3.14)$$

where the $d \times d$ diagonal matrix⁶ $\hat{\tau}_{\hat{\mu}\hat{\nu}}$ is given by:

$$\hat{\tau}_{\hat{\mu}\hat{\nu}} \equiv \frac{1}{2} [\mathcal{D}_{\hat{\mu}\hat{\nu}}(\epsilon) + \mathcal{D}_{\hat{\nu}\hat{\mu}}(-\epsilon)]. \quad (3.15)$$

Expressing \mathcal{E} in terms of the real vielbein E ($\mathcal{E} = SE$, with S given in eq. (2.3)) we can go to the real basis, where eq. (3.14) reads:

$$\exp \left\{ \frac{\pi i}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} {}^t \hat{m}_{\hat{\mu}} \left[{}^t E \sqrt{\left(1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2\right) \left(1 - (\mathcal{F}_{\hat{\nu}}^{(d)})^2\right)} \begin{pmatrix} \hat{\tau}^S & i\hat{\tau}^A \\ -i\hat{\tau}^A & \hat{\tau}^S \end{pmatrix}_{\hat{\mu}\hat{\nu}} E \right] \hat{m}_{\hat{\nu}} \right\}, \quad (3.16)$$

where $\hat{\tau}^S$ and $\hat{\tau}^A$ are the symmetric and the antisymmetric part of $\hat{\tau}$, in the exchange of $\hat{\mu}$, $\hat{\nu}$. As $\hat{\tau}$ is purely imaginary and $Im \hat{\tau}^S$ is positive definite because of the Riemann bilinear identities [38] and moreover $\left(\sqrt{1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2}\right)_{ab} = \delta_{ab} |1 - \mathcal{F}_{\hat{\mu}}^{(d)}|_{aa}$, the convergence of the series in eq. (3.5) is assured.

Following [39] we can rewrite the Born-Infeld square roots above in yet another way by using another important consequence of the Riemann bilinear identities [38]:

$$C_{\hat{\mu}\hat{\nu}} = C_{\hat{\nu}\hat{\mu}} \equiv \frac{1}{2} [\mathcal{D}_{\hat{\mu}\hat{\nu}}(\epsilon) - \mathcal{D}_{\hat{\nu}\hat{\mu}}(-\epsilon)] = i \frac{\sin(\pi\epsilon_{\hat{\mu}}) \sin(\pi\epsilon_{\hat{\nu}} - \pi\epsilon_g)}{\sin(\pi\epsilon_g)}, \quad \hat{\nu} \geq \hat{\mu}, \quad (3.17)$$

where also $C_{\hat{\mu}\hat{\nu}}$ and the sines are $d \times d$ diagonal matrices whose entries depend on the different values of ϵ . This will be useful for the analysis of the degeneration limit in the next section. The $2d \times 2d$ matrix $\text{diag} \{\sin(\pi\epsilon_{\mu}), \sin(\pi\epsilon_{\mu})\}$ can be written as:⁷

$$\begin{pmatrix} \sin(\pi\epsilon_{\mu}) & 0 \\ 0 & \sin(\pi\epsilon_{\mu}) \end{pmatrix} = \sqrt{\frac{1}{1 - (\mathcal{F}_{\mu}^{(d)})^2}}. \quad (3.18)$$

⁶Our $\hat{\tau}$ is equal to the τ of [38]; we indicate it with a different symbol in order to avoid confusion with the standard period matrix.

⁷We will not keep track of the sign choices for the square roots: they clearly depend on whether each ϵ_{μ} is negative or positive.

This can be checked by rewriting the sine in terms of exponentials which are directly related to the components of \mathcal{S} in the complex basis: $\sin(\pi\epsilon_\mu^\alpha) = [\sqrt{2 - \mathcal{S}_\mu^{-1} - \mathcal{S}_\mu}]_{\alpha\alpha}/2$, $\alpha = 1, 2, \dots, d$. Also, using the same procedure, we have

$$\begin{pmatrix} \sin(\pi\epsilon_{\hat{\nu}} - \pi\epsilon_g) & 0 \\ 0 & \sin(\pi\epsilon_{\hat{\nu}} - \pi\epsilon_g) \end{pmatrix} = \frac{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} (\mathcal{F}_g^{(d)} - \mathcal{F}_{\hat{\nu}}^{(d)})}{\sqrt{(1 - (\mathcal{F}_g^{(d)})^2) (1 - (\mathcal{F}_{\hat{\nu}}^{(d)})^2)}}, \quad (3.19)$$

with $\mathcal{F}_\mu^{(d)}$ defined as in eq. (2.7). From (3.18) and (3.19) one can see that

$$C_{\hat{\mu}\hat{\nu}} = C_{\hat{\nu}\hat{\mu}} = \frac{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathcal{F}_{\hat{\nu}}^{(d)} - \mathcal{F}_g^{(d)})}{\sqrt{(1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2) (1 - (\mathcal{F}_{\hat{\nu}}^{(d)})^2)}}, \quad \text{for } \hat{\nu} \geq \hat{\mu}. \quad (3.20)$$

Thus we can eliminate the square roots in (3.14) in favor of C ; then it is convenient to decompose $\hat{\tau}$ into its symmetric ($\hat{\tau}^S$) and the antisymmetric ($\hat{\tau}^A$) parts. Hence we can use eq. (2.7) to rewrite the diagonal fields $\mathcal{F}^{(d)}$ in terms of the \mathcal{F} 's, take advantage of the identity $\mathcal{F}_\mu - \mathcal{F}_\nu = F_\mu - F_\nu$, insert the contribution of the cocycles given by eq. (3.10) and finally get the classical contribution to the twisted partition function describing wrapped D-branes on an generic T^{2d} :

$$\begin{aligned} \mathcal{A} = \sum \Delta \exp \left\{ \pi i \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} \sum_{M < N} {}^t \hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}_{\hat{\nu}}^N \right\} e^{2\pi i \sum_{\hat{\mu}=1}^{g-1} {}^t \rho_{\hat{\mu}} \hat{m}_{\hat{\mu}}} \\ \times \exp \left\{ \frac{\pi i}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} {}^t \hat{m}_{\hat{\mu}} {}^t \mathcal{E} \left[\begin{pmatrix} \frac{\hat{\tau}^S}{C} & 0 \\ 0 & -\frac{\hat{\tau}^S}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} + \begin{pmatrix} \frac{\hat{\tau}^A}{C} & 0 \\ 0 & \frac{\hat{\tau}^A}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} \right] {}^t \mathcal{E}^{-1} F_{\hat{\mu}\hat{\nu}} \hat{m}_{\hat{\nu}} \right\}. \end{aligned} \quad (3.21)$$

where $F_{\hat{\mu}\hat{\nu}}$ is the matrix defined in eq. (3.11).⁸ Then the full partition function is simply given by (3.4), where $\mathcal{A}^{(0)}$ is substituted by the complete expression above.

If we restrict ourselves to the case of a factorizable torus $T^{2d} = (T^2)^d$, then (3.21) agrees⁹ with the results of [38]. In order to make contact with their setup it is first useful to perform a T-Duality in such a way that the singled-out boundary with identifications encoded in the R_0 matrix is transformed back to a magnetized D-Brane. From (2.49) we can compute the transformation of the difference between two gauge field strengths

$$F'_i - F'_j = (F_j {}^t c + {}^t d)^{-1} (F_j - F_i) (c F_i - d)^{-1}. \quad (3.22)$$

In particular the duality we performed to put $R_0 = -1$ had $d = c F_0$. Then by combining eq. (3.22) with the transformation of the winding numbers (2.22), the amplitude (3.21)

⁸The inverse of C must be meant only with respect to the Lorentz indices, at fixed $\hat{\mu}$ and $\hat{\nu}$.

⁹Apart from some factors of two.

reduces to the following product of terms, each one related to a single T^2 :

$$\begin{aligned} \mathcal{A}_{(T^2)^d}^w &= \sum \Delta \exp \left\{ \pi i \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} \sum_{M < N} t \hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}_{\hat{\nu}}^N \right\} e^{2\pi i \sum_{\hat{\mu}=1}^{g-1} t \rho_{\hat{\mu}} \hat{m}_{\hat{\mu}}} \quad (3.23) \\ &\times \prod_{\alpha=1}^d \exp \left\{ \frac{\pi}{2} \sum_{\hat{\mu}, \hat{\nu}=1}^{g-1} t \hat{m}_{\hat{\mu}}^\alpha \left[\left(\frac{\tau_S(\epsilon^\alpha)}{C(\epsilon^\alpha)} \right)_{\hat{\mu}\hat{\nu}} \mathcal{I}^{(\alpha)} + i \left(\frac{\tau_A(\epsilon^\alpha)}{C(\epsilon^\alpha)} \right)_{\hat{\mu}\hat{\nu}} \right] F_{\hat{\mu}\hat{\nu}}^{(\alpha)} \hat{m}_{\hat{\nu}}^\alpha \right\}, \end{aligned}$$

where now

$$\begin{aligned} F_{\hat{\mu}\hat{\nu}} &= F_{\hat{\nu}\hat{\mu}} = (F_0 - F_{\hat{\mu}})(F_0 - F_g)^{-1}(F_g - F_{\hat{\nu}}) \quad , \quad \text{for } \hat{\nu} \geq \hat{\mu}, \quad (3.24) \\ t \rho_{\hat{\mu}} \hat{m}_{\hat{\mu}} &= [C_{\hat{\mu}} - C_g(F_0 - F_g)^{-1}(F_0 - F_{\hat{\mu}}) - C_0(F_0 - F_g)^{-1}(F_{\hat{\mu}} - F_g)] \hat{m}_{\hat{\mu}} \end{aligned}$$

and $\mathcal{I}^{(\alpha)}$ is the complex structure of each of the T^2 's defined as eq. (2.9). This can be compared with eq. (A.28) of [38]. The norm of the vector v_i appearing there is related to the Born-Infeld square roots: $|v_i U| / \sqrt{U_2 T_2} = w_{\hat{\mu}} \sqrt{(1 - (\mathcal{F}_{\hat{\mu}}^{(d)})^2)}$. One can use this in (3.20) and (4.8) together with the explicit expression for the T^2 complex structure (4.7) to check that the two results are related by a further T-duality that exchanges $T \leftrightarrow -1/U$.

4. Twist fields couplings on T^{2d}

In this section we will focus on the vacuum diagram with three boundaries (i.e. $g = 2$) and study the degeneration limit where all three open string propagators in figure 1b become long and thin. In this situation the partition function factorizes in two tree-level 3-point correlators between twist fields. This result provides the main contribution for the computation of the Yukawa couplings in string phenomenological models and of other terms in the effective action generated by stringy instantons [17, 18]. For $g = 2$ the only non-vanishing entry for $\hat{\tau}$ is clearly $\hat{\tau}_{11}^S$, which has been computed, at leading order in this degeneration limit, in [3]. Again the novelty of the present computation resides in the analysis of the classical part (3.21), while all other ingredients of the partition function (3.4) factorize exactly as before, since they do not depend on the wrapping numbers or the Wilson lines. In the degeneration limit under study we have [3] that $\mathcal{D}_{11}(\epsilon) \rightarrow 0$ from which

$$\left(\frac{\hat{\tau}^S(\epsilon)}{C(\epsilon)} \right)_{11} \rightarrow -1_{d \times d} \quad (4.1)$$

thus the exponential in the second line of eq. (3.21) becomes

$$\exp \left\{ \frac{\pi}{2} t \hat{m}_1 \mathcal{I}(F_2 - F_1) \hat{m}_1 \right\}, \quad (4.2)$$

where we have introduced the complex structure of the torus in the integral basis as in eq. (2.9). In order to write the final form of the amplitude as a sum over unconstrained integers, it is necessary to solve the conservations (3.6), which, in the case under study, become

$$\hat{n}_1 = -F_1 \hat{m}_1 \quad , \quad \hat{n}_2 = -F_2 \hat{m}_2 \quad , \quad \hat{m}_1 + \hat{m}_2 = 0 \quad . \quad (4.3)$$

Of course the solutions must have integer Kaluza-Klein and windings numbers, so there must exist a minimal¹⁰ integer invertible matrix H such that $F_1 H$ and $F_2 H$ are integer matrices and the solution can be written as $\hat{m}_1 = Hh$, with $h \in \mathbb{Z}^{2d}$. Then we define $\mathcal{I}' = {}^t H \mathcal{I} {}^t H^{-1}$, which is still a complex structure, and $F \equiv {}^t H (F_2 - F_1) H$; so the degeneration limit of the amplitude (3.21) in our $g = 2$ case is

$$\mathcal{A} = \sum_{h \in \mathbb{Z}^{2d}} \exp \left\{ \frac{\pi}{2} \left[{}^t h \mathcal{I}' F h + 2i \sum_{M < N} h^M F_{MN} h^N \right] \right\} \times \exp \{ 2\pi i {}^t \rho_1 I h \} , \quad (4.4)$$

with a possible half-integer shift of ρ_1 (see the appendix for further details).

By unitarity it must be possible to rewrite the result in (4.4) as a sum where each term is the product of two functions that are one the complex conjugate of the other. Each function represents the classical contribution to the coupling among three twist fields, while the quantum contribution follows from the factorization of the other terms in the partition functions (3.4), see [3]. The presence of the sum is due to the fact that the vacuum describing an open string stretched between two magnetized D-branes has a finite degeneracy [5] in compact spaces. So each string state has a number of replica and the various terms in the sum describe the couplings between these different copies of the twist fields (of course this is exactly what we need, in phenomenological models, to describe the Yukawa couplings for different families). Let us see how this works in the simple case of the 2-torus.

4.1 The two-torus example

For a generic tilted torus the metric can be written as a function of the Kähler and complex structure moduli

$$G = \frac{T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} = {}^t \mathcal{E} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{E}, \quad (4.5)$$

having defined the complex structure as $U = U_1 + iU_2$ and the Kähler form as $T = T_1 + iT_2$. Thus the vielbein (2.4) reads

$$\mathcal{E} = \sqrt{\frac{T_2}{2U_2}} \begin{pmatrix} 1 & U \\ 1 & \bar{U} \end{pmatrix} \quad (4.6)$$

from which one can write the explicit form of the T^2 complex structure in the integral basis

$$\mathcal{I} = {}^t \mathcal{E} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} {}^t \mathcal{E}^{-1} = -\frac{1}{U_2} \begin{pmatrix} U_1 & -1 \\ |U|^2 & -U_1 \end{pmatrix}, \quad (4.7)$$

The magnetic fields on the two magnetized D-Branes are identified by the Chern numbers p_i and the products of the wrappings along the two cycles of the torus W_i :

$$F_i = \begin{pmatrix} 0 & \frac{p_i}{W_i} \\ -\frac{p_i}{W_i} & 0 \end{pmatrix}, \quad i = 1, 2. \quad (4.8)$$

¹⁰By minimal we mean that any other matrix with the same property is an integer multiple of H .

One can easily check that the matrix H is simply proportional to the 2×2 identity, namely $H = W_1 W_2 / \delta \otimes 1_{2 \times 2}$, where $\delta = \text{G.C.D.} \{W_1, W_2\}$. This implies that eq. (4.4) can be put in the following form¹¹

$$\sum_{h_1, h_2 \in \mathbb{Z}} \exp \left\{ -\frac{\pi}{2} \frac{I}{U_2} [h_1^2 + |U|^2 h_2^2 + 2U h_1 h_2] + 2\pi i \frac{1}{\delta} \mathcal{C}_M h^M \right\}, \quad (4.9)$$

with

$$I = \frac{W_1^2 W_2^2}{\delta^2} \left(\frac{p_2}{W_2} - \frac{p_1}{W_1} \right) = I_{21} \frac{W_1 W_2}{\delta^2}, \quad (4.10)$$

where we introduced the intersection numbers

$$I_{ij} = p_i W_j - p_j W_i \quad (4.11)$$

and

$$\mathcal{C}_M = W_1 W_2 [(F_1 - F_2) Y_0 + (C_1 - C_2)]_M. \quad (4.12)$$

We want to perform a T-Duality in such a way that also the zero-th D-Brane is magnetized. The intersection numbers are invariant under this operation, while the form of the matrix I is modified due to the transformation of the wrapping numbers. Recalling that the T-Duality relating a Dirichlet to a magnetized brane is encoded in an $O(2, 2, \mathbb{Z})$ matrix of the type in eq. (2.17) with¹²

$$d = \begin{pmatrix} 0 & -p_0 \\ p_0 & 0 \end{pmatrix} \quad \text{and} \quad c = \tilde{w}_0 \equiv \tilde{W}_0 \otimes 1_{2 \times 2}, \quad (4.13)$$

it is possible to show, combining the invariance of I_{21} with eq. (3.22), that $W_1 \rightarrow I_{01}$ and $W_2 \rightarrow I_{20}$. Thus

$$I = \frac{I_{01} I_{21} I_{20}}{\delta^2} \quad (4.14)$$

with $\delta = \text{G.C.D.} \{I_{20}, I_{10}\} = \text{G.C.D.} \{I_{20}, I_{10}, I_{21}\}$, since we can make use of the property $I_{21} \tilde{W}_0 + I_{20} \tilde{W}_1 + I_{01} \tilde{W}_2 = 0$. Under the same duality the open string moduli transform as follows

$$C_\mu = \frac{\tilde{C}_\mu}{\tilde{w}_0 (F_0 - F_\mu)} \quad \text{and} \quad Y_0 = \tilde{w}_0 \tilde{C}_0 \quad (4.15)$$

hence

$$\mathcal{C}_M = \tilde{W}_0 I_{12} \tilde{C}_M^{(0)} + \tilde{W}_1 I_{20} \tilde{C}_M^{(1)} + \tilde{W}_2 I_{01} \tilde{C}_M^{(2)}. \quad (4.16)$$

where the superscript (i) indicates the three boundaries and the subscript M is the Lorentz index.

The configurations studied [3] had δ and all \tilde{W}_i equal to 1. In this case I is always an even number, since it is the product of three integers that sum to zero. Thus the contribution of the cocycles in (4.4) is irrelevant and (4.9) can be rewritten as it was done

¹¹In the configuration of [3] one gets $I > 0$, otherwise one should write $|I|$, because of the note before eq. (3.18).

¹²We indicate with a tilde the quantities in the picture with a magnetized zero-th D-Brane.

in [3]. We choose not to do that here, because it is easier to deal always with (4.9) without treating the case $\tilde{W}_i = 1$ separately. In order to factorize the amplitude above and find the Yukawa couplings corresponding to the states of the open strings stretched between pairs of D-Branes with different magnetic fields on their world volume, it is necessary to first perform a Poisson resummation on the integer h_1

$$\sum_{h_1=-\infty}^{+\infty} e^{-\pi A h_1^2 + 2\pi h_1 A s} = \frac{1}{\sqrt{A}} e^{\pi A s^2} \sum_{h_1=-\infty}^{+\infty} e^{-\pi \frac{h_1^2}{A} - 2\pi i h_1 s}, \quad A > 0. \quad (4.17)$$

This yields a new form of the same amplitude which is easy to factorize once we introduce a new pair of integers, r and k , through the relations

$$h_1 = rI + l = I \left(r + \frac{l}{I} \right) \quad \text{and} \quad h_2 = k - r, \quad (4.18)$$

where $l = 1, \dots, I$. It is manifest that in this way both the former and the latter pair of integers range in the whole \mathbb{Z} . Notice that this is ensured by summing over the additional integer l as well. Simple algebraic manipulations then lead to the product of two Jacobi Theta-functions as follows

$$\mathcal{A} = \sqrt{\frac{2U_2}{I}} \sum_{l=1}^I \vartheta \left[\begin{matrix} \frac{l}{I} - \frac{1}{I} \frac{c_1}{\delta} \\ \frac{c_2}{\delta} \end{matrix} \right] (0|IU) \times \vartheta \left[\begin{matrix} \frac{l}{I} - \frac{1}{I} \frac{c_1}{\delta} \\ -\frac{c_2}{\delta} \end{matrix} \right] (0| -I\bar{U}) \quad (4.19)$$

This result generalizes the one of ref. [3] and is in agreement¹³ with the section 3.1.3 of [12], as we find that, if the intersection numbers I_{ij} are not coprime, one has $I_{20}I_{01}I_{21}/\delta^2$ non vanishing Yukawa couplings, labeled by the integer l . Indeed, in the dual picture involving intersecting D-Branes, every open string living in the intersection between two fixed D-Branes, say for instance i and j , will only couple to $|I_{jk}I_{ik}|/\delta^2$ strings from the intersections between the D-Brane k and the D-Branes i and j [12].

4.2 Twist fields couplings on a generic T^{2d} compactification

In order to factorize the classical contribution to the partition function (4.4) in the most general case analyzed in section 3 and read the corresponding twist field couplings, we need to use the properties of the complex structure. Let us first decompose \mathcal{I}' in terms of its $d \times d$ blocks

$$\mathcal{I}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (4.20)$$

Hence it is simple to check that $\mathcal{I}'^2 = -1$ yields

$$AB = -BD, \quad \text{and} \quad A^2 = -(1 + BC). \quad (4.21)$$

¹³The apparent mismatch related to the presence of the wrapping numbers of the D-Branes in the Wilson lines dependence of the couplings is resolved by checking that $W_i C_1^{(i)}, W_i C_2^{(i)} \in [0, 1]$ as well as the parameters ϵ_i and θ_i defined in [12].

Then we choose a basis for the torus lattice where the combination ${}^tH(F_2 - F_1)H$ takes the following form

$$F = \begin{pmatrix} 0 & \hat{F} \\ -\hat{F} & 0 \end{pmatrix}. \quad (4.22)$$

This can be done by putting the matrix F in the form of eq. (2.35) (thanks again to the result of the appendix) and by a suitable relabeling of the rows and the columns. Notice that this relabeling does not affect the form of the amplitude to be factorized. For sake of brevity, we also introduce the $2d$ -components vector $\beta = {}^tH\rho_1$. Then the general amplitude to be factorized has the following form¹⁴

$$\sum_{h_i \in \mathbb{Z}^d} \exp \left\{ \frac{\pi}{2} \left[({}^th_1 \ {}^th_2) \begin{pmatrix} -B\hat{F} & (i+A)\hat{F} \\ (i-D)\hat{F} & C\hat{F} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right] + 2\pi i {}^t\beta_1 h_1 + 2\pi i {}^t\beta_2 h_2 \right\}. \quad (4.23)$$

As next step we need to perform a Poisson resummation

$$\sum_{h_1 \in \mathbb{Z}^d} e^{-\pi {}^th_1 B h_1 + 2\pi {}^th_1 B s} = \frac{1}{\sqrt{\text{Det} B}} e^{\pi {}^t s B s} \sum_{h_1 \in \mathbb{Z}^d} e^{-\pi {}^th_1 B^{-1} h_1 - 2\pi i {}^th_1 s} \quad (4.24)$$

on the first d components of h which we indicate with h_1 . So in our case we have

$$B = \frac{1}{2} \hat{F} {}^tB \quad \text{and} \quad s = \frac{1}{2} B^{-1} (A + i) \hat{F} h_2 + i B^{-1} \beta_1. \quad (4.25)$$

The first exponential in the r.h.s. of the Poisson resummation formula (4.24) yields a quadratic term in the vector h_2 , which can be combined with a similar contribution present in the initial expression (4.23)

$$\begin{aligned} \frac{\pi}{4} {}^th_2 \hat{F} (i + {}^tA) {}^tB^{-1} (i + A) \hat{F} h_2 + \frac{\pi}{2} {}^th_2 C \hat{F} h_2 &= \\ &= \frac{\pi}{2} {}^th_2 \left[\hat{F} (i + {}^tA) \hat{F}^{-1} B^{-1} (i + A) \hat{F} + C \hat{F} \right] h_2. \end{aligned} \quad (4.26)$$

Recalling that $\mathcal{I}'\hat{F}$ is a symmetric matrix, it is not difficult to see that $A\hat{F} = -\hat{F}{}^tD$. Some algebraic manipulations involving these identities simplify the previous expression into

$$i\pi {}^th_2 B^{-1} (i + A) \hat{F} h_2. \quad (4.27)$$

Hence the Poisson resummation performed on eq. (4.23) gives

$$\begin{aligned} \frac{1}{\sqrt{\text{Det}(2B\hat{F})}} \sum_{h_i \in \mathbb{Z}^d} \exp \left\{ i\pi \left[{}^th_2 B^{-1} (i + A) \hat{F} h_2 + 2i {}^t\beta_1 {}^tB^{-1} \hat{F}^{-1} \beta_1 + \right. \right. \\ \left. \left. + {}^th_2 \hat{F} (i + {}^tA) \hat{F}^{-1} B^{-1} \beta_1 + {}^t\beta_1 \hat{F}^{-1} B^{-1} (i + A) \hat{F} h_2 \right. \right. \\ \left. \left. + 2 {}^t\beta_2 h_2 + 2i {}^th_1 {}^tB^{-1} \hat{F}^{-1} h_1 - 2 {}^th_2 B^{-1} (i + A) h_1 - \right. \right. \\ \left. \left. 4i {}^th_1 {}^tB^{-1} \hat{F}^{-1} \beta_1 \right] \right\}. \end{aligned} \quad (4.28)$$

¹⁴In the following i indicates a $d \times d$ imaginary matrix: $i \equiv i 1_{d \times d}$.

In the following manipulations we will focus on the expression in the square brackets only. It is useful to observe that the by redefining

$$\gamma_1 = \hat{F}^{-1}\beta_1 \quad \text{and} \quad k = \hat{F}^{-1}h_1 \quad (4.29)$$

and making use again of the identities mentioned earlier, involving the entries of the complex structure and of \hat{F} , one can rewrite the content of the square brackets above as

$$\begin{aligned} & {}^t h_2 B^{-1}(i+A)\hat{F}h_2 + 2i {}^t \gamma_1 B^{-1}\hat{F}\gamma_1 + {}^t h_2 B^{-1}(i+A)\hat{F}\gamma_1 + \\ & + {}^t \gamma_1 B^{-1}(i+A)\hat{F}h_2 + 2 {}^t \beta_2 h_2 + 2i {}^t k B^{-1}\hat{F}k - \\ & - 2 {}^t h_2 B^{-1}(i+A)\hat{F}k - 4i {}^t k B^{-1}\hat{F}\gamma_1 . \end{aligned} \quad (4.30)$$

As k in general is no longer a column of integers, it is convenient to write it distinguishing its integer part from the remainder

$$k = r + \hat{F}^{-1}l, \quad (4.31)$$

with $r \in \mathbb{Z}^d$ and $l_\alpha \in [1, \hat{F}_{\alpha\alpha}]$. Thus the expression above reads

$$\begin{aligned} & 2 {}^t (r + \hat{F}^{-1}l - \gamma_1) i B^{-1}\hat{F} (r + \hat{F}^{-1}l - \gamma_1) + {}^t h_2 B^{-1}(i+A)\hat{F}h_2 + \\ & - 2 {}^t h_2 B^{-1}(i+A)\hat{F} (r + \hat{F}^{-1}l - \gamma_1) + 2 {}^t \beta_2 h_2 . \end{aligned} \quad (4.32)$$

Finally, defining $s = r - h_2 \in \mathbb{Z}^d$, one has

$$\begin{aligned} & {}^t (r + \hat{F}^{-1}l - \gamma_1) B^{-1}(i-A)\hat{F} (r + \hat{F}^{-1}l - \gamma_1) + 2 {}^t (r + \hat{F}^{-1}l - \gamma_1) \beta_2 + \\ & {}^t (s + \hat{F}^{-1}l - \gamma_1) B^{-1}(i+A)\hat{F} (s + \hat{F}^{-1}l - \gamma_1) - 2 {}^t (s + \hat{F}^{-1}l - \gamma_1) \beta_2 . \end{aligned} \quad (4.33)$$

Thus the factorized amplitude reads

$$\begin{aligned} \mathcal{A} = \sum_{l_\alpha=1}^{\hat{F}_{\alpha\alpha}} \frac{1}{\sqrt{\text{Det}(2B\hat{F})}} & \vartheta \left[\begin{array}{c} \hat{F}^{-1}(l - \beta_1) \\ \beta_2 \end{array} \right] \left(0 \middle| B^{-1}(i-A)\hat{F} \right) \times \\ & \vartheta \left[\begin{array}{c} \hat{F}^{-1}(l - \beta_1) \\ -\beta_2 \end{array} \right] \left(0 \middle| B^{-1}(i+A)\hat{F} \right) \end{aligned} \quad (4.34)$$

written in terms of d -dimensional Theta-functions

$$\vartheta \left[\begin{array}{c} a \\ b \end{array} \right] (\nu|\tau) = \sum_{h \in \mathbb{Z}^d} \exp [\pi i {}^t (h+a)\tau(h+a) + 2\pi i {}^t (\nu+b)(h+a)] . \quad (4.35)$$

Notice that the function in the second line of the eq. (4.34) is indeed the complex conjugate of the one in the first line, as \hat{F} , A and B are real $d \times d$ matrices since \mathcal{I} in eq. (2.9) is real. This function is to be interpreted as the classical contribution to the Yukawa couplings for three twisted states arising in a generic T^{2d} compactification of string theory with magnetised space filling D-Branes. The sum in front of the couplings reveals their

multiplicity, given by $\text{Det}\hat{F} = \prod_{\alpha=1}^d \hat{F}_{\alpha\alpha}$. Finally for the sake of completeness let us write the expression for the correlator between three twist fields (fixing one l , i.e. one particular coupling) including also its quantum contribution with the convention $\epsilon_0^\alpha + \epsilon_1^\alpha + \epsilon_2^\alpha = 1$:¹⁵

$$\langle \sigma_{\epsilon_1} \sigma_{\epsilon_2} \sigma_{\epsilon_3} \rangle = \prod_{i=1}^3 \prod_{\alpha=1}^d \left[\frac{\Gamma(1 - \epsilon_i^\alpha)}{\Gamma(\epsilon_i^\alpha)} \right]^{\frac{1}{4}} (\text{Det}(2B))^{-\frac{1}{4}} \times \vartheta \left[\begin{matrix} \hat{F}^{-1}(l - \beta_1) \\ \beta_2 \end{matrix} \right] \left(0 \mid B^{-1}(i - A)\hat{F} \right). \quad (4.36)$$

We can check that this result is in agreement with the literature considering in particular the multiplicity of the Yukawa couplings in the case of parallel fluxes, i.e. when all of the boundaries are magnetized D-Branes with magnetic fields of the type (2.35) put in the form (4.22). Notice that this setup can always be T-dualized into a configuration of intersecting D-Branes on a $2d$ -dimensional torus that is not geometrically a direct product of d T^2 's. However in the counting of the non vanishing 3-point correlators between twisted states the metric of the torus is not involved, and thus we expect this multiplicity is equal to the one already calculated [12] in fully factorizable models of D-Branes at angles on $(T^2)^d$. Indeed following the steps of the previous subsection it is not difficult to see that $H = w_1 w_2 / \delta$, δ being a diagonal matrix whose eigenvalues are the G.C.D.'s of the entries of w_1 and w_2 as in eq. (2.43). Then by generalizing also the definition (4.11) into a diagonal matrix, with the same structure and the intersection numbers as entries, one has $\hat{F} = I_{21} w_1 w_2 / \delta^2$. Upon the straightforward generalization of the T-Duality in eq. (4.13) $\hat{F} \rightarrow I_{21} I_{20} I_{01} / \delta^2$ where δ now contains the G.C.D.'s of the entries of the three intersection numbers and

$$\text{Det}\hat{F} = \prod_{\alpha=1}^d \frac{(p_{2\alpha} W_{0\alpha} - p_{0\alpha} W_{2\alpha})(p_{0\alpha} W_{1\alpha} - p_{1\alpha} W_{0\alpha})(p_{2\alpha} W_{1\alpha} - p_{2\alpha} W_{2\alpha})}{\delta_{\alpha\alpha}^2} \quad (4.37)$$

This number agrees with the product of the multiplicity of Yukawa couplings in each of the T^2 's inside the T^{2d} defined by the form of the magnetic fields (see [12]).

Acknowledgments

We wish to thank Giulio Bonelli, Paolo Di Vecchia, Francisco Morales, Igor Pesando, Sanjaye Ramgoolam, Alessandro Tanzini and Steve Thomas for useful discussions and comments. This work is partially supported by the European Community's Human Potential under contract MRTN-CT-2004-005104 and MRTN-CT-2004-512194, and by the Italian MIUR under contract PRIN 2005023102. R. R. and S. Sc. thank the Galileo Galilei Institute for Theoretical Physics, and S. Sc. thanks the Queen Mary University, for hospitality during the completion of this work.

¹⁵We use the notations of section 4 of [3].

A. Generic and block-diagonal integer matrices

A generic integer $2d \times 2d$ antisymmetric matrix M can always be put in a block-diagonal form by means of a transformation of the type $M \rightarrow {}^tOMO$, O being a unimodular integer matrix.

In order to show that this is indeed the case one can observe that any antisymmetric matrix:

$$M = \begin{pmatrix} A & B \\ -{}^tB & C \end{pmatrix} \tag{A.1}$$

where A and C are $2k \times 2k$ and $2(d - k) \times 2(d - k)$ antisymmetric matrices and B is a rectangular $2k \times 2(d - k)$ matrix, can be block diagonalized by

$$O = \begin{pmatrix} 1_{2k} & -A^{-1}B \\ 0 & 1_{2(d-k)} \end{pmatrix}, \tag{A.2}$$

getting:

$${}^tOMO = \begin{pmatrix} A & 0 \\ 0 & {}^tBA^{-1}B + C \end{pmatrix} \tag{A.3}$$

Notice that each block of tOMO is also an antisymmetric matrix and that the determinant of the matrix O is one. Since the form of tOMO in eq. (A.3) is independent of the choice of k , one can use an iterative procedure to obtain the final block diagonal form always choosing $k = 1$. Using this procedure $d - 1$ times one finds

$${}^tOMO = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_2 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_d \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.4}$$

where: $O = O_1O_2 \dots O_{d-1}$. By a suitable permutation of the rows and of the columns of the matrix above one can rewrite the transformed matrix as

$$M' = \begin{pmatrix} 0 & \tilde{M} \\ -\tilde{M} & 0 \end{pmatrix}, \tag{A.5}$$

where $\tilde{M} = \text{diag}\{a_1, a_2, \dots, a_d\}$.

If the matrix M has integer elements, the transformed matrix will be integer only if $\text{Det}A = 1$. We will now show that it is always possible to reduce to this case at any step of the iterative procedure. Obviously, if at least one element of the matrix M is 1, it is enough to suitably relabel the rows and the columns. Otherwise, if $\text{Det}A \neq 1$ we can distinguish various cases. The simplest one is when in a row two elements are coprime. By relabeling of the rows and the columns it is possible to put these elements (a and b) in the first row as follows:

$$M = \begin{pmatrix} 0 & a & b & \cdots \\ -a & 0 & c & \cdots \\ -b & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{A.6}$$

Then it is easy to see that the unimodular integer matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & x & -b & 0 & \cdots \\ 0 & y & a & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{A.7})$$

with x, y solution of the Diophantine equation $ax + by = 1$, transforms M into:

$$M \rightarrow {}^tQM = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & c & \cdots \\ 0 & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{A.8})$$

which has $\text{Det}A = 1$.

If, instead, there is no row with two coprime elements, but in eq. (A.6) $a \neq \pm b$ and their greatest common denominator d is different from 1, one can repeat the previous step with the matrix:

$$Q' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & x & -\frac{b}{d} & 0 & \cdots \\ 0 & y & \frac{a}{d} & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{A.9})$$

where now x and y solve the Diophantine equation $ax + by = d$. This yields

$$M \rightarrow {}^tQ'M = \begin{pmatrix} 0 & d & 0 & \cdots \\ -d & 0 & c & \cdots \\ 0 & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (\text{A.10})$$

One has to apply this procedure (which does not change the other elements of the first row, from the 4th column on) till the matrix is reduced to one of the following cases: either in the first row of the transformed M there are two coprime non-vanishing entries, then one can use the matrix Q to obtain $a_1 = 1$; or all of the non-zero elements there coincide with $\pm d$. This is the case if for instance in the original matrix M the first row contained elements which were all multiples of d . If there is any other row in the transformed matrix with two different non-zero elements $a \neq \pm b$ for which d is not a divisor, then, by exchanging rows and columns among themselves, it is possible to bring this as the first row and reapply the transformations encoded in Q or Q' . Otherwise one can have two possible forms for the transformed matrix. One possibility is that all of the elements of M are integer multiples of d . In this case the common divisor d can be factored out to reduce to the case with $\text{Det}A = 1$.

The other possibility is that the matrix has diagonal blocks, in which all the elements are multiple of different integers d_i . If all the non trivial blocks are 2×2 , we have got our aim; otherwise we can factor out d_i from the block of larger dimensions; for each of them we are again reduced to the case with $\text{Det}A = 1$. Repeating the procedure, if needed, we finally end to the matrix:

$$M_1 = \begin{pmatrix} 0 & d_1 & 0 & 0 & \cdots \\ -d_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & d_2 & \cdots \\ 0 & 0 & -d_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A.11})$$

with all integer elements.

Although this already is the final form we are after, for sake of completeness we recall that the normal form of the initial matrix (A.4), as discussed in [35], has the further property that $a_{\alpha+1}/a_\alpha \in \mathbb{N}, \forall \alpha$. In order to achieve this (even if it is not strictly necessary for the computations considered here) one can use the following transformation

$$Q'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A.12})$$

that mixes the d_i 's and gives back a form that can be reduced by means of either Q or Q' . Following this procedure, it is possible to convince oneself that the final matrix (A.4) entries satisfy the property mentioned above, since one actually ends the repeated application of Q , Q' , and Q'' only if in the first 2×2 block there is a one, or if the matrix is proportional to an integer as a whole.

Coming back to our main problem, we can now apply the procedure outlined in this appendix to rewrite the quantized magnetic field (2.27) on a generic magnetized D-Brane in the form (2.35). It is first convenient to define an integer matrix P associated to the magnetic field (2.27)

$$P = F \times \text{m.c.m} \{w_M w_N, M \neq N, \forall M, N = 1, \dots, 2d\} = \omega F. \quad (\text{A.13})$$

where ω is the minimum common multiple of all the pairs of wrappings that appear in the denominators of eq. (2.27). The integer matrix P can now be transformed into a block-diagonal form as in eq. (A.4) by means of an integer unimodular matrix O that preserves the lattice of the torus as

$$P \rightarrow {}^t O P O = \omega {}^t O F O = \omega F_{\text{block}}. \quad (\text{A.14})$$

Hence F_{block} will have the same form as eq. (A.4) with rational entries whose numerators and denominators could still have factors in common. By expurgating these factors one exactly recovers the form in eq. (2.35). First of all we will show that the phase factor in the

boundary state (2.47) is not affected by the change of the fundamental cell in the lattice torus performed in (A.14). It reads:

$$\text{Ph} = \exp \left[i\pi \sum_{M < N} \hat{m}^M F_{MN} \hat{m}^N \right] = \exp \left[\frac{i\pi}{\omega} \sum_{M < N} \hat{m}^M P_{MN} \hat{m}^N \right] \quad (\text{A.15})$$

where P is the integer matrix defined in (A.13), that we write in the form of eq. (A.1). To write it as a block diagonal matrix, we use the techniques just discussed; focusing at first on the simplest case with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A.16})$$

let us consider how the phase Ph of (A.15) transforms under the substitution: $\hat{m} = O\hat{m}'$, with O as in eq. (A.2). One gets:

$$\begin{aligned} \text{Ph} = \exp \left[\frac{i\pi}{\omega} \left(\hat{m}'_1 A_{12} \hat{m}'_2 + \sum_{j=3}^{2d} B_{2j} \hat{m}'_j A_{12} \hat{m}'_2 - \hat{m}'_1 A_{12} \sum_{j=3}^{2d} B_{1j} \hat{m}'_j \right. \right. \\ \left. \left. - \sum_{j,k=3}^{2d} B_{2j} \hat{m}'_j A_{12} B_{1k} \hat{m}'_k + \sum_{j=3}^{2d} \hat{m}'_1 B_{1j} \hat{m}'_j + \sum_{j,k=3}^{2d} B_{2j} \hat{m}'_j B_{1k} \hat{m}'_k \right. \right. \\ \left. \left. + \sum_{j=3}^{2d} \hat{m}'_2 B_{2j} \hat{m}'_j - \sum_{j,k=3}^{2d} B_{1j} \hat{m}'_j B_{2k} \hat{m}'_k + \sum_{k>j=3}^{2d} \hat{m}'_j C_{jk} \hat{m}'_k \right) \right] \quad (\text{A.17}) \end{aligned}$$

By remembering that all the winding numbers \hat{m}^N must be integer multiples of the corresponding wrapping numbers w_N , one can check that, in spite of the denominator ω , all the terms in the exponent are integer multiples of $i\pi$. In fact combining the form of the matrices (A.2) and (A.13), one finds the following expressions for $\hat{m} = O\hat{m}'$

$$\begin{aligned} \hat{m}_1 &= \hat{m}'_1 + \frac{\omega}{w_2} \sum_{i=3}^{2d} p_{2i} \frac{\hat{m}'_i}{w_i} \\ \hat{m}_2 &= \hat{m}'_2 - \frac{\omega}{w_1} \sum_{i=3}^{2d} p_{1i} \hat{m}'_i \\ \hat{m}'_i &= \hat{m}_i \end{aligned} \quad (\text{A.18})$$

In this case, in order to have the matrix A in the form (A.16), it is necessary that $p_{12} = 1$ and $\omega = w_1 w_2$, hence, since from the last line of the previous equation \hat{m}'_i/w_i must be integer, it is also true, in the first and second line, that the transformed winding numbers \hat{m}'^N are integer multiples of the wrapping numbers w_N , $\forall N = 1, 2, \dots, 2d$. Thus each of the terms in the sum (A.17) is an integer number, as one can check for instance considering the first term of the second line in eq. (A.17); writing $\hat{m}'^N = m'^N w_N$ with $m'^N \in \mathbb{Z}$ one gets:

$$\frac{1}{\omega} B_{2j} \hat{m}'_j A_{12} B_{1k} \hat{m}'_k = \frac{1}{\omega} \frac{\omega p_{2j}}{w_2 w_j} m'_j w_j \frac{\omega p_{1k}}{w_1 w_k} m'_k w_k = p_{2j} m'_j p_{1k} m'_k \in \mathbb{Z}. \quad (\text{A.19})$$

So we can freely change the sign of each term in eq. (A.17), obtaining

$$\text{Ph} = \exp \left[i\pi \left(\sum_{M < N} \hat{m}'^M ({}^tOFO)_{MN} \hat{m}'^N + \frac{1}{\omega} \sum_{j=3}^{2d} B_{1j} B_{2j} \hat{m}'_j{}^2 \right) \right] \quad (\text{A.20})$$

$$= \exp \left[i\pi \left(\sum_{M < N} \hat{m}'^M ({}^tOFO)_{MN} \hat{m}'^N + \sum_{j=3}^{2d} p_{1j} p_{2j} \frac{\hat{m}'_j}{w_j} \right) \right], \quad (\text{A.21})$$

where we have used the explicit expression of B_{1j} and B_{2j} in terms of the Chern numbers p_{1j} and p_{2j} and of the winding numbers; moreover we have taken into account the fact that $(m'_j)^2$ has the same parity (even/odd) as $m'_j = \hat{m}'_j/w_j$. Thus the phase factor can be written in terms of the transformed field tOFO and of the transformed winding numbers \hat{m}' 's with the same functional form as the original one (A.15), with a half-integer shift of the Wilson line when $p_{1j}p_{2j}$ is odd.

If tOFO is already block diagonal, we have ended our job, otherwise we have to repeat the procedure. In an analogous fashion, if the entries of A are not equal to one, one can check that the transformations related to the matrices in eq. (A.7) and (A.9), involved in reducing A to the form considered in the previous example, also preserve the form of the phase factor up to half-integer Wilson lines. With similar manipulations it is also possible to prove, in a basis in which $(F_2 - F_1)$ is block-diagonal, that the phase factors in eq. (4.4) follow from those in eq. (4.2). As usual, one has to introduce $h \in \mathbb{Z}^{2d}$ by using $\hat{m}_1 = Hh$; then it is possible to check that the combination $\sum_{M < N} (Hh)^M (F_2 - F_1)_{MN} (Hh)^N$ is equal, modulus two, to $\sum_{M < N} h^M [H(F_2 - F_1)H]_{MN} h^N$, apart from terms quadratic in h^M that can be reabsorbed into a half-integer shift of the Wilson lines.

Finally we mention that the transformations discussed in this appendix do not affect the other contributions to the amplitude, in the effective field theory limit that we consider for the factorization, if one suitably redefines the complex structure in eq. (4.2). Indeed the combination ${}^t\hat{m}_1 \mathcal{I} F \hat{m}_1$ can be rewritten, by redefining $\hat{m}_1 = O\hat{m}'_1$, as

$${}^t\hat{m}'_1 {}^tO \mathcal{I} F O \hat{m}'_1 = {}^t\hat{m}'_1 {}^tO \mathcal{I} O^{-1} {}^tO F O \hat{m}'_1 = {}^t\hat{m}'_1 \hat{\mathcal{I}} F_{\text{block}} \hat{m}'_1, \quad (\text{A.22})$$

$\hat{\mathcal{I}}$ still being a good complex structure and F_{block} being in the form (A.4).

References

- [1] C. Lovelace, *Pomeron form-factors and dual Regge cuts*, *Phys. Lett.* **B 34** (1971) 500.
- [2] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)].
- [3] R. Russo and S. Sciuto, *The twisted open string partition function and Yukawa couplings*, *JHEP* **04** (2007) 030 [[hep-th/0701292](#)].
- [4] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The Conformal Field Theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.
- [5] A. Abouelsaood, J. Callan, G. Curtis, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

- [6] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys.* **B 480** (1996) 265 [[hep-th/9606139](#)].
- [7] F. Marchesano, *Progress in D-brane model building*, *Fortschr. Phys.* **55** (2007) 491 [[hep-th/0702094](#)].
- [8] T.T. Burwick, R.K. Kaiser and H.F. Muller, *General Yukawa couplings of strings on $z(n)$ orbifolds*, *Nucl. Phys.* **B 355** (1991) 689.
- [9] J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, *Higher twisted sector couplings of $Z(n)$ orbifolds*, *Nucl. Phys.* **B 397** (1993) 379 [[hep-th/9207049](#)].
- [10] S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski, *Yukawa couplings for bosonic $Z(n)$ orbifolds: their moduli and twisted sector dependence*, *Mod. Phys. Lett.* **A 7** (1992) 3059 [[hep-th/9204037](#)].
- [11] S. Stieberger, *Moduli and twisted sector dependence on $Z(n) \times Z(m)$ orbifold couplings*, *Phys. Lett.* **B 300** (1993) 347 [[hep-th/9211027](#)].
- [12] D. Cremades, L.E. Ibáñez and F. Marchesano, *Yukawa couplings in intersecting D-brane models*, *JHEP* **07** (2003) 038 [[hep-th/0302105](#)].
- [13] M. Cvetič and I. Papadimitriou, *Conformal Field Theory couplings for intersecting D-branes on orientifolds*, *Phys. Rev.* **D 68** (2003) 046001 [[hep-th/0303083](#)].
- [14] S.A. Abel and A.W. Owen, *Interactions in intersecting brane models*, *Nucl. Phys.* **B 663** (2003) 197 [[hep-th/0303124](#)].
- [15] S.A. Abel and A.W. Owen, *N -point amplitudes in intersecting brane models*, *Nucl. Phys.* **B 682** (2004) 183 [[hep-th/0310257](#)].
- [16] D. Lüst, P. Mayr, R. Richter and S. Stieberger, *Scattering of gauge, matter and moduli fields from intersecting branes*, *Nucl. Phys.* **B 696** (2004) 205 [[hep-th/0404134](#)].
- [17] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua — The seesaw mechanism for D-brane models*, *Nucl. Phys.* **B 771** (2007) 113 [[hep-th/0609191](#)].
- [18] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [[hep-th/0609213](#)].
- [19] M. Bianchi and E. Trevigne, *The open story of the magnetic fluxes*, *JHEP* **08** (2005) 034 [[hep-th/0502147](#)].
- [20] L. Álvarez-Gaumé, C. Gomez, G.W. Moore and C. Vafa, *Strings in the operator formalism*, *Nucl. Phys.* **B 303** (1988) 455.
- [21] P. Di Vecchia et al., *N point g loop vertex for a free bosonic theory with vacuum charge q* , *Nucl. Phys.* **B 322** (1989) 317.
- [22] J. Polchinski, *String theory*, Cambridge University Press, Cambridge U.K. (1998).
- [23] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [[hep-th/9401139](#)].
- [24] I. Pesando, *Boundary states for branes with non trivial homology in constant closed and open background*, [hep-th/0505052](#).

- [25] P. Di Vecchia, A. Liccardo, R. Marotta, F. Pezzella and I. Pesando, *Boundary state for magnetized D9 branes and one-loop calculation*, [hep-th/0601067](#).
- [26] P. Di Vecchia and A. Liccardo, *D-branes in string theory. i*, *NATO Adv. Study Inst. Ser. C. Math. Phys. Sci.* **556** (2000) 1 [[hep-th/9912161](#)].
- [27] P. Di Vecchia and A. Liccardo, *D-branes in string theory. II*, [hep-th/9912275](#).
- [28] C.G. Callan Jr., C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221.
- [29] P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando and F. Pezzella, *Wrapped magnetized branes: two alternative descriptions?*, [arXiv:0709.4149](#).
- [30] Z. Guralnik and S. Ramgoolam, *Torons and D-brane bound states*, *Nucl. Phys.* **B 499** (1997) 241 [[hep-th/9702099](#)].
- [31] Z. Guralnik and S. Ramgoolam, *From 0-branes to torons*, *Nucl. Phys.* **B 521** (1998) 129 [[hep-th/9708089](#)].
- [32] G. 't Hooft, *Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus*, *Commun. Math. Phys.* **81** (1981) 267.
- [33] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, [hep-th/9602052](#).
- [34] A. Hashimoto, *Perturbative dynamics of fractional strings on multiply wound D-strings*, *Int. J. Mod. Phys. A* **13** (1998) 903 [[hep-th/9610250](#)].
- [35] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons Inc. (1994).
- [36] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [[hep-th/9910053](#)].
- [37] C. Bachas and M. Porrati, *Pair creation of open strings in an electric field*, *Phys. Lett.* **B 296** (1992) 77 [[hep-th/9209032](#)].
- [38] I. Antoniadis, K.S. Narain and T.R. Taylor, *Open string topological amplitudes and gaugino masses*, *Nucl. Phys.* **B 729** (2005) 235 [[hep-th/0507244](#)].
- [39] M. Bianchi and E. Trevigne, *Gauge thresholds in the presence of oblique magnetic fluxes*, *JHEP* **01** (2006) 092 [[hep-th/0506080](#)].